From Lie crossed modules to tensor hierarchies PSU 12-3-2022

Jim Stasheff (with Sylvain Lavau)

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Thoughts

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"And your hair has become very white ;

And yet you incessantly ....

Do you think, at your age, it is right ?"

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Nothing is more fruitful - all mathematicians know it - than those obscure analogies, those disturbing reflections of one theory in another ; those furtive caresses, those inexplicable discords ; nothing also gives more pleasure to the researcher. The day comes when the illusion dissolves ; the yoked theories reveal their common source before disappearing.

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Barry Mazur

There are metaphorical bridges that connect subjects and viewpoints *cajoling* us to view one field from the perspective of another.

Motivation

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Classical gauge theories : gauge fields A can be regarded as 1-forms on a manifold M with values in a representation V of a *gauge* Lie algebra $\mathfrak{g}$:

$$A \in \Omega^1(M; V)$$

Supergravity theories : In what are known as supergravity field theories and others, techniques from classical theories do not behave as desired ;

The magic phrase is :

the field strengths do not transform *covariantly*.

That is, the transform of a certain field ϕ may not be proportional to ϕ, i.e. not in an ideal generated by ϕ, but rather the transform contains some unwanted terms.

To paraphrase what physicist do, they add more fields. In particular, to compensate for this failure, they add 2-forms $B \in \Omega^2(M, W)$ taking values in a $\mathfrak{g}$-**module** W_2 and a linear map $\partial_{-1} : W_2 \longrightarrow V$ to kill the obstruction/discrepancy to covariance.

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But then there occurs a 2-form obstruction which is a ∂_{-1} cocycle , so they add a 3-form $C \in \Omega^3(M, X)$ taking values in a $\mathfrak{g}$-**module** W_3. Then again there is a failure of covariance ... and so on. Thus there are further fields which are forms with values in the a sequence of $\mathfrak{g}$-modules W_i.

This is the essence of a *tensor hierarchy*.

Cast of Characters

- Lie crossed module aka differential crossed module
- Leibniz algebra
- embedding tensor
- tensor hierarchy

A *Lie crossed module* consists of a pair of Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}_{-1}$ equipped with two Lie algebra homomorphisms

$\partial : \mathfrak{g}_{-1} \to \mathfrak{g}_0$ and $\triangleright : \mathfrak{g}_0 \to Der(\mathfrak{g}_{-1})$,

the Lie algebra of Lie derivations,

better written as

$$x \triangleright b \text{ for } x \in \mathfrak{g}_0,\ b \in \mathfrak{g}_{-1}$$

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A Lie crossed module is equivalently a dgLa (differential graded Lie algebra) $(\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}, \partial, [\,.\,,.\,])$:

- $[\,.\,,.\,]\big|_{\mathfrak{g}_0 \wedge \mathfrak{g}_0} = [\,.\,,.\,]_{\mathfrak{g}_0}$,
- $[\,.\,,.\,]\big|_{\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1}} = 0$,
- $[\,.\,,.\,]\big|_{\mathfrak{g}_0 \wedge \mathfrak{g}_{-1}} =$ action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$,

A (left) *Leibniz algebra* is a vector space V together with a bilinear operation $\circ : V \otimes V \to V$ satisfying the relation :

$$x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z).$$

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For V Leibniz, the adjoint map $\mathrm{ad} : V \longrightarrow \mathrm{End}(V), x \longmapsto x \circ -$ is :

- a derivation of $\circ$:

$$\mathrm{ad}_x(y \circ z) = \mathrm{ad}_x(y) \circ z + y \circ \mathrm{ad}_x(z))$$

- a morphism of Leibniz algebras : for $x, y \in V$,

$$\mathrm{ad}_{x \circ y} = [\mathrm{ad}_x, \mathrm{ad}_y]$$

Embedding tensor

Let V be a representation of a Lie algebra $\mathfrak{g}$ via $\rho : \mathfrak{g} \to \mathrm{End}(V)$, hence with a binary operation

$$\circ : V \otimes V \to V.$$

An *embedding tensor* is a **lift** of ad, the adjoint for $\circ$:

$$\begin{tikzcd} & \mathfrak{g} \arrow[d, "\rho"] \\ V \arrow[ur, "\Theta"] \arrow[r, "\mathrm{ad}"] & \mathrm{End}(V) \end{tikzcd}$$

satisfying the **quadratic constraint** :

$$\Theta(\Theta(x) \cdot y) = [\Theta(x), \Theta(y)]_{\mathfrak{g}}$$

Consequences

Consequences :

- V is a Leibniz algebra :

$$x \circ y = \Theta(x) \cdot y$$

- $\mathfrak{h} = \mathrm{Im}(\Theta)$ is a **Lie subalgebra of $\mathfrak{g}$**
-
- Θ is $\mathfrak{h}$-equivariant,
  but NOT necessarily $\mathfrak{g}$-equivariant.

Supergravity

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Tensor hierarchies

In certain physics papers, this leads to what physicists call a *tensor hierarchy*,

i.e. a tower of $\mathfrak{g}$-modules forming a **chain complex**, in which the successive p-form gauge fields $A, B, C, D, \ldots$ take values :

$$\ldots \xrightarrow{\partial_{-3}} T_{-3} \xrightarrow{\partial_{-2}} T_{-2} \xrightarrow{\partial_{-1}} V \xrightarrow{\Theta} \mathfrak{g}$$

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In particular cases considered in the physics literature, this chain complex possesses a *differential graded Lie algebra (dgLa)* structure containing physically relevant information

Goal

Objective of the talk : build an appropriate purely mathematical functor

$$\{\text{Lie-Leibniz triple}\} \longrightarrow \{\text{dgLa}\}$$

Strategy : build the tower of spaces step by step after making a wise choice for the first step.

Our construction has been widely influenced by (Greitz et al., 2014 ; Cederwall & Palmkvist, 2015).

What is an embedding tensor ?

For V Leibniz, the adjoint map $\mathrm{ad} : V \longrightarrow \mathrm{End}(V), x \longmapsto x \circ -$ is :

- a derivation of $\circ$:

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- a morphism of Leibniz algebras

$$\mathrm{ad}_{x \circ y} = [\mathrm{ad}_x, \mathrm{ad}_y]$$

Assume that V is a representation of a Lie algebra $\mathfrak{g}$ via $\rho : \mathfrak{g} \to \mathrm{End}(V)$. An *embedding tensor* is a **lift** of ad :

$$\begin{tikzcd} & \mathfrak{g} \arrow[d, "\rho"] \\ V \arrow[ur, "\Theta"] \arrow[r, "\mathrm{ad}"'] & \mathrm{End}(V) \end{tikzcd}$$

What is a Lie-Leibniz triple ?

Definition

A *Lie-Leibniz triple* is a triple $(\mathfrak{g}, V, \Theta)$ where

- $\mathfrak{g}$ is a Lie algebra,
- V is a $\mathfrak{g}$-**module**,
- $\Theta : V \longrightarrow \mathfrak{g}$ is a linear map called the *embedding tensor*, satisfying the **quadratic constraint** :

$$\Theta(\Theta(x) \cdot y) = [\Theta(x), \Theta(y)]_{\mathfrak{g}}$$

Hierarchy

Recall $\mathfrak{h} \subset \mathfrak{g}$.

Theorem

From a Lie-Leibniz triple $V \xrightarrow{\Theta} \mathfrak{g}$, we can build a dgLa :

$$\dots \xrightarrow{\partial_{-3}} T_{-3} \xrightarrow{\partial_{-2}} T_{-2} \xrightarrow{\partial_{-1}} V \xrightarrow{\Theta} \mathfrak{h} \longrightarrow 0$$

extending the Lie crossed module

$$V \xrightarrow{\Theta} \mathfrak{h} \longrightarrow 0$$

Construction

The Leibniz product $\circ$ can be split in two parts :

$$[x,y] = \frac{1}{2}\big(x \circ y - y \circ x\big) \qquad \text{and} \qquad \{x,y\} = \frac{1}{2}\big(x \circ y + y \circ x\big)$$

so that : $x \circ y = [x,y] + \{x,y\}$

The skew-symmetric bracket $[\,.\,,.\,]$ does not satisfy the Jacobi identity :

$$\big[x,[y,z]\big]+\big[y,[z,x]\big]+\big[z,[x,y]\big] = -\frac{1}{3}\Big(\big\{x,[y,z]\big\}+\big\{y,[z,x]\big\}+\big\{z,[x,y]\big\}\Big)$$

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Let the *ideal of squares* be $\mathcal{I} = \mathrm{Span}\big(x\circ x = \{x,x\} \;\big|\; x\in V\big)$

The properties of the embedding tensor imply the inclusions :

$$\mathcal{I} \underbrace{\subset}_{(a)} \mathrm{Ker}(\Theta) \underbrace{\subset}_{(b)} \mathrm{Ker}(\mathrm{ad})$$

(a) $\Theta(x\circ y) = [\Theta(x),\Theta(y)]_{\mathfrak{g}}$ (b) $\mathrm{ad}_x(y) = \Theta(x)\cdot y$

Construction of T_{-2}

In supergravity examples, one notices that the ideal of squares $\mathcal{I}$ is generated by the 2-forms B. More precisely :

- gauge fields 1-forms A span $T_{-1} = V$
- gauge fields 2-forms B span some $\mathfrak{g}$-**module** T_{-2} such that $\boxed{\partial_{-1} : T_{-2} \longrightarrow V \text{ is onto } \mathcal{I}}$

♦ *if $\mathcal{I}$ is not a $\mathfrak{g}$-module* $\Longrightarrow T_{-2} \neq \mathcal{I}$

♦ Idea : lift $\{\,.\,,.\,\} : S^2V \twoheadrightarrow \mathcal{I}$

♦ $\mathrm{Ker}\big(\{\,.\,,.\,\}\big) \subset S^2(V)$ is an $\mathfrak{h}$-module, but may NOT be a $\mathfrak{g}$-module.

First step of the construction

We define K_{-2} to be the **biggest** $\mathfrak{g}$-sub-module of $\mathrm{Ker}\big(\{\,.\,,.\,\}\big)$, and

$$T_{-2} = S^2(V)\Big/K_{-2}$$

$$\begin{tikzcd}
 & T_{-2} \arrow[d, two heads, "\partial_{-1}"] \\
S^2(V) \arrow[ur, "q_{-2}"] \arrow[r, two heads, "\{\,.\,,.\,\}"] & \mathcal{I} \arrow[d, two heads, "\Theta"] \\
 & \mathfrak{h} \subset \mathfrak{g}
\end{tikzcd}$$

Construction of T_{-3}

Let $F_{-1} = V$ (in degree -1) and let $F_\bullet = \bigoplus_{i\geq 1} F_{-i}$ be the free graded Lie algebra generated by F_{-1}.

$$F_{-2} = \wedge^2(F_{-1}) \simeq S^2(V), \quad \wedge^3(F_{-1}) = S^3(V), \quad V \otimes S^2(V) \simeq \wedge^3(F_{-1}) \oplus F_{-3}$$

$$\begin{tikzcd}
0 \arrow[r] & V \otimes K_{-2} \arrow[r, "\mathrm{id}"] \arrow[d] & V \otimes K_{-2} \arrow[r] \arrow[d, dashed] & 0 \\
S^3(V) \arrow[r] & V \otimes S^2(V) \arrow[r] \arrow[d, "\mathrm{id}\otimes q_{-2}"] & F_{-3} \arrow[r] \arrow[d] & 0 \\
S^3(V) \arrow[r] & V \otimes T_{-2} \arrow[r, "q_{-3}"] & F_{-3}/K_{-3} \arrow[r] & 0
\end{tikzcd}$$

Exactness of the second row implies exactness of the third, so we set

$$\boxed{T_{-3} = F_{-3}/K_{-3}}$$

Construction of $T_{-(n+1)}$

Suppose that all the $T_{-i} = {}^{F_{-i}}\big/_{K_{-i}}$ have been built up to order n, where $K_{-i} \subset F_{-i}$ is a $\mathfrak{g}$-submodule.

$$\begin{tikzcd}
 & 0 \arrow[r] & \bigoplus_{j=1}^{n-1} F_{-j} \otimes K_{-(n+1)+j} \arrow[r, "\mathrm{id}"] \arrow[d] & \bigoplus_{j=1}^{n-1} F_{-j} \otimes K_{-(n+1)+j} \arrow[r] \arrow[d, dashed] & 0 \\
 & \Lambda^3(F_\bullet)\big|_{-(n+1)} \arrow[r] & \Lambda^2(F_\bullet)\big|_{-(n+1)} \arrow[r] \arrow[d] & F_{-(n+1)} \arrow[r] \arrow[d] & 0 \\
 & \Lambda^3\big(F_\bullet\big)\big|_{-(n+1)} \arrow[r] & \Lambda^2\Big(\bigoplus_{i=1}^{n} T_{-i}\Big)\Big|_{-(n+1)} \arrow[r, "q_{-(n+1)}"] & {}^{F_{-(n+1)}}\big/_{K_{-(n+1)}} \arrow[r] & 0
\end{tikzcd}$$

We define $\boxed{T_{-(n+1)} = {}^{F_{-(n+1)}}\big/_{K_{-(n+1)}}}$.

The end of the hierarchy

Continuing the induction provides us with a (possibly infinite) **graded vector space** $T_\bullet = \bigoplus_{i=1}^{\infty} T_{-i}$, the quotient of $F_\bullet$ by the graded ideal $K_\bullet$.

It has the following properties :

- Every vector space T_{-i} is a $\mathfrak{g}$-module ;
- every map $q_{-i} : \wedge^2 T_\bullet|_{-i} \twoheadrightarrow T_{-i}$ is $\mathfrak{g}$-equivariant ;
- $T_{-1} = V$ (in degree -1) ;
- $T_{-i} = 0$ for every $i \geq 2$ **if and only if** $(V, \circ)$ is a Lie algebra.

$T_\bullet$ can be equipped with a graded Lie algebra structure with bracket :

$$\boxed{q = [\![\, . \, , . \,]\!] : \wedge^2 T_\bullet \longrightarrow T_\bullet}$$

where $q|_{\wedge^2 T_\bullet|_{-i}}$ is the quotient map $q_{-i} : \wedge^2 T_\bullet|_{-i} \twoheadrightarrow T_{-i}$.

What about physical implications ?

♦ In some “physical“ examples, it is known that the graded Lie bracket of the tensor hierarchy contains *all relevant physical information* on the field strengths and the gauge transformations (Greitz et al., 2013 ; Bonezzi & Hohm, 2019)

♦ Beyond its mathematical interest *per se*, the construction has promising applications in giving a better understanding of higher gauge theories in e.g. *double* and in *exceptional field theory*, as well as any forthcoming Leibniz gauge theory.

♦ Some differential graded algebras of *fields* and *gauge parameters* can be extended to include *equations of motion* and *Noether identities*. Are these related to tensor hierarchies ?

And beyond ?

⧫ The (differential) graded Lie algebra structure on the tensor hierarchy we have constructed is different from that in (Palmkvist, 2013) and (Palmkvist & Cederwall, 2015). It remains to check if the two constructions *coincide*.

⧫ How about generalizing the construction to Lie-Leibniz algebroids ?

⧫ How about applying tensor hierarchies to foliations ?

with Sylvain Lavau, arXiv :2003.07838v4

Thank You