Higher representations and Heegaard-Floer theory II

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Monoidal category for $\mathfrak{gl}(1|1)^{>0}$

(Joint work with Andrew Manion)

$$\mathfrak{gl}(1|1)^{>0} = \mathbb{C}_{odd} \ , \quad U(\mathfrak{gl}(1|1)^{>0}) = \mathbb{C}[e]/e^2$$

Naive : Monoidal cat $\mathcal{U}_{naive}$ with two objects $1, E$, $E \otimes E = 0$

Khovanov : $\mathcal{U}$ strict monoidal $\Bbbk$-linear cat generated by an object E and a map $\tau : E^2 \to E^2$ with relations $\tau^2 = 0$ and

$$\begin{tikzcd}
 & E^3 \arrow[dl, "E\tau"'] \arrow[dr, "\tau E"] & \\
E^3 \arrow[d, "\tau E"'] & \circlearrowright & E^3 \arrow[d, "E\tau"] \\
E^3 \arrow[dr, "E\tau"'] & & E^3 \arrow[dl, "\tau E"] \\
 & E^3 &
\end{tikzcd}$$

char $\Bbbk = 2$. $\mathcal{U}$ differential cat. with $d(\tau) = 1$

objects : E^n, $n \geq 0$. $\mathrm{Hom}(E^n, E^m) = 0$ if $m \neq n$.

$$\mathrm{End}(E^n) \xleftarrow{\sim} H_n = \Bbbk\langle T_1, \dots, T_{n-1} \mid T_i^2 = 0, T_iT_j = T_jT_i \text{ if } |i-j|>1, T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \rangle$$

$$\begin{matrix} E^{i-1} & E^2 & E^{n-i-1} \\ \downarrow \mathrm{id} & \downarrow \tau & \downarrow \mathrm{id} \\ E^{i-1} & E^2 & E^{n-i-1} \end{matrix} \longleftarrow\!\shortmid \ T_i$$

$$d(T_i) = 1$$
$$\Downarrow$$
$$H^*(H_n) = 0 \quad \text{if } n \geq 2$$

So $\boxed{H^*(\mathcal{U}) = \mathcal{U}_{naive}}$

Def | A 2-representation on a differential cat. $\mathcal{V}$ is a diff. monoidal functor $\mathcal{U} \to \mathrm{End}(\mathcal{V})$

Same as data of $E : \mathcal{V} \to \mathcal{V}$, $\tau \in \mathrm{End}(E^2)$ satisfying $\tau^2 = 0$ and braid relation

2-repr. are the objects of a 2-category : * a 1-arrow $(\mathcal{V}, E, \tau) \to (\mathcal{V}', E', \tau')$ is the data of $\Phi : \mathcal{V} \to \mathcal{V}'$ diff functor and $\varphi : \Phi E \xrightarrow{\sim} E' \Phi$ (with $d(\varphi) = 0$)

such that

$$\begin{tikzcd} \Phi E^2 \arrow[r] \arrow[d, "\Phi\tau"'] \arrow[dr, phantom, "\circlearrowright"] & E'\Phi E \arrow[r] & E'^2 \Phi \arrow[d, "\tau'\Phi"] \\ \Phi E^2 \arrow[r] & E'\Phi E \arrow[r] & E'^2\Phi \end{tikzcd}$$

* 2-arrows

$$\begin{tikzcd} \mathcal{V} \arrow[r, bend left=50, "(\Phi_1, \varphi_1)"{name=U}] \arrow[r, bend right=50, "(\Phi_2, \varphi_2)"'{name=D}] & \mathcal{V}' \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd}$$

$\alpha : \Phi_1 \to \Phi_2$ natural transformation

such that

$$\begin{tikzcd} \Phi_1 E \arrow[r, "\varphi_1"] \arrow[d, "\alpha E"'] \arrow[dr, phantom, "\circlearrowright"] & E'\Phi_1 \arrow[d, "E'\alpha"] \\ \Phi_2 E \arrow[r, "\varphi_2"'] & E'\Phi_2 \end{tikzcd}$$

Assume E has a left adjoint $E^\vee$. Have $\tau \in \mathrm{End}(E^2) \xrightarrow{\sim} \mathrm{End}((E^2)^\vee) = \mathrm{End}((E^\vee)^2)$

$(\mathcal{V}, E^\vee, \tau)$: left dual. Similarly: if E has a right adjoint, get right dual.

<u>Examples</u>

- $\mathcal{V} =$ differential k-vect. spaces, $E = \tau = 0$ "trivial 2-repr."
- $\mathcal{V} = \mathcal{U}$, $E = e \otimes -$
- $\mathcal{V} = \mathcal{U}$, $E = - \otimes e$
- $\mathcal{V} = \mathcal{U}$ is a bimodule 2-repr: it is acted on by $\mathcal{U} \otimes \mathcal{U}$

<u>Lax bimod 2-rep</u> on $\mathcal{V}$: $(E_1, \tau_1), (E_2, \tau_2)$ 2-rep and $\sigma : E_2 E_1 \to E_1 E_2$ such that

$$\begin{tikzcd}
E_2^2 E_1 \arrow[r, "E_2\sigma"] \arrow[d, "\tau_2 E_1"'] & E_2 E_1 E_2 \arrow[r, "\sigma E_2"] \arrow[d, phantom, "\circlearrowright"] & E_1 E_2^2 \arrow[d, "E_1\tau_2"] \\
E_2^2 E_1 \arrow[r, "E_2\sigma"'] & E_2 E_1 E_2 \arrow[r, "\sigma E_2"'] & E_1 E_2^2
\end{tikzcd}
\qquad
\begin{tikzcd}
E_2 E_1^2 \arrow[r, "\sigma E_1"] \arrow[d, "E_2\tau_1"'] & E_1 E_2 E_1 \arrow[r, "E_1\sigma"] \arrow[d, phantom, "\circlearrowright"] & E_1^2 E_2 \arrow[d, "\tau_1 E_2"] \\
E_2 E_1^2 \arrow[r] & E_1 E_2 E_1 \arrow[r] & E_1^2 E_2
\end{tikzcd}$$

Coproduct

Tensor structure on 2-rep?

$$\begin{tikzcd}
\text{2-rep} \times \text{2-rep} \arrow[rr, "\otimes"] & & \text{2-rep} \\
\text{2-birep} \arrow[u, phantom, "\cap" description] & & \\
\text{2-laxbirep} \arrow[u, phantom, "\cap" description] \arrow[rruu, "\Delta"'] & &
\end{tikzcd}$$

$(\mathcal{W}, E_1, \tau_1, E_2, \tau_2, \sigma)$ a lax bimod 2-rep.

* Define $\Delta\mathcal{W}$ a differential cat.

objects : pairs (m, π), $m \in$ idempotent completion of pretriangulated closure of $\mathcal{W}$

$\pi \in \mathrm{Hom}(E_2(m), E_1(m))$, $d(\pi) = 0$

such that

$$\begin{tikzcd}
E_2^2(m) \arrow[r, "E_2\pi"] \arrow[d, "\tau_2"'] & E_2E_1(m) \arrow[r, "\sigma"] & E_1E_2(m) \arrow[r, "E_1\pi"] \arrow[d, phantom, "\circlearrowright"] & E_1^2(m) \arrow[d, "\tau_1"] \\
E_2^2(m) \arrow[r, "E_2\pi"'] & E_2E_1(m) \arrow[r, "\sigma"'] & E_1E_2(m) \arrow[r, "E_1\pi"'] & E_1^2(m)
\end{tikzcd}$$

$$\mathrm{Hom}((m,\pi),(m',\pi')) = \left\{ f \in \mathrm{Hom}(m,m') \;\middle|\; \begin{tikzcd} E_2(m) \arrow[r, "\pi"] \arrow[d, "E_2 f"'] \arrow[dr, phantom, "\circlearrowright"] & E_1(m) \arrow[d, "E_1 f"] \\ E_2(m') \arrow[r, "\pi'"'] & E_1(m') \end{tikzcd} \right\}$$

* Define $E : \mathcal{D}(W) \to \mathcal{D}(W)$ by

$$E(m,\pi) = \left(\mathrm{Cone}\left(E_2(m) \xrightarrow{\pi} E_1(m)\right), \begin{pmatrix} \sigma \circ E_2\pi \circ \tau_2 & \sigma \\ 0 & \tau_1 \circ E_1\pi \circ \sigma \end{pmatrix} \right)$$

$$\begin{tikzcd} E_2^2(m) \arrow[r, phantom, "\oplus"] \arrow[rr, bend left, "E_2\pi"] \arrow[d, "\sigma \circ E_2\pi \circ \tau_2"'] & & E_2E_1(m) \arrow[dll, "\sigma"'] \arrow[d, "\tau_1 \circ E_1\pi \circ \sigma"] \\ E_1E_2(m) \arrow[r, phantom, "\oplus"] \arrow[rr, bend right, "E_1\pi"'] & & E_1^2(m) \end{tikzcd}$$