

# Higher representations and Heegaard-Floer theory II

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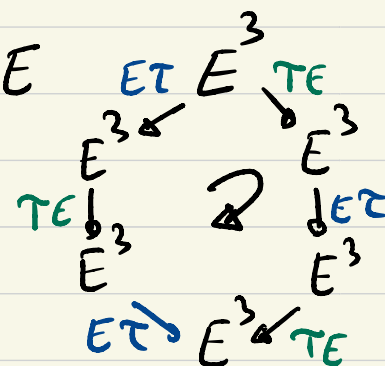
# Monoidal category for $gl(1|1)^{>0}$

(Joint work with Andrew Manion)

$$gl(1|1)^{>0} = \mathbb{C}_{\text{odd}}, \quad U(gl(1|1)^{>0}) = \mathbb{C}[e]/e^2$$

Naive: Monoidal cat  $\mathcal{U}_{\text{naive}}$  with two objects  $1, E$ ,  $E \otimes E = 0$

Khovanov:  $\mathcal{U}$  strict monoidal  $\mathbb{R}$ -linear cat. generated by an object  $E$  and a map  $\tau: E^2 \rightarrow E^1$  with relations  $\tau^2 = 0$  and



char  $\mathbb{k} = 2$ .  $\mathcal{U}$  differential cat. with  $d(\tau) = 1$

objects:  $E^n, n \geq 0$ .  $\text{Hom}(E^n, E^m) = 0$  if  $m \neq n$ .

$$\text{End}(E^n) \xleftarrow{\sim} H_n = \mathbb{k}\langle \tau_1, \dots, \tau_{n-1} \mid \tau_i^2 = 0, \tau_i \tau_j = \tau_j \tau_i \text{ if } |i-j| > 1, \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \rangle$$

$$\begin{array}{ccc} E^{i-1} & E^2 & E^{n-i-1} \\ \text{id} \downarrow & \tau & \text{id} \downarrow \\ E^{i-1} & E^2 & E^{n-i-1} \end{array} \longleftarrow \tau_i$$

$$d(\tau_i) = 1$$

$$H^*(H_n) = 0 \text{ if } n \geq 2$$

So

$$H^*(\mathcal{U}) = \mathcal{U}_{\text{naive}}$$

Def | A 2-representation on a differential cat.  $\mathcal{V}$  is a diff. monoidal functor  $\mathcal{U} \rightarrow \text{End}(\mathcal{V})$

Same as data of  $E: \mathcal{V} \rightarrow \mathcal{V}$ ,  $\tau \in \text{End}(E^2)$  satisfying  $\tau^2 = 0$  and braid relation

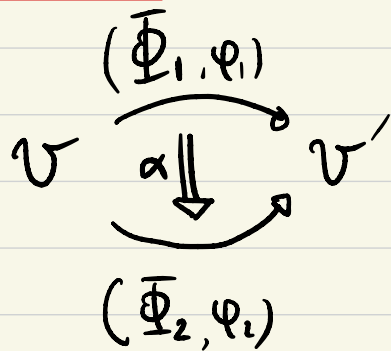
2-repr. are the objects of a 2-category: a 1-arrow  $(\mathcal{V}, E, \tau) \rightarrow (\mathcal{V}', E', \tau')$

is the data of  $\Phi: \mathcal{V} \rightarrow \mathcal{V}'$  diff. functor and  $\varphi: \Phi E \xrightarrow{\sim} E' \Phi$  (with  $d(\varphi) = 0$ )

such that

$$\begin{array}{ccccc} \Phi E^2 & \longrightarrow & E' \Phi E & \longrightarrow & E'^2 \Phi \\ \downarrow \Phi \tau & \curvearrowright & & & \downarrow \tau' \Phi \\ \Phi E^2 & \longrightarrow & E' \Phi E & \longrightarrow & E'^2 \Phi \end{array}$$

\* 2-arrows



$\alpha: \Phi_1 \rightarrow \Phi_2$  natural transformation

such that

$$\begin{array}{ccc} \Phi_1 E & \xrightarrow{\varphi_1} & E' \Phi_1 \\ \alpha E \downarrow & \curvearrowright & \downarrow E' \alpha \\ \Phi_2 E & \xrightarrow{\varphi_2} & E' \Phi_2 \end{array}$$

Assume  $E$  has a left adjoint  $E^\vee$ . Have  $\tau \in \text{End}(E^2) \xrightarrow{\sim} \text{End}((E^2)^\vee) = \text{End}((E^\vee)^2)$

$(\mathcal{U}, E^\vee, \tau)$ : left dual. Similarly: if  $E$  has a right adjoint, get right dual.

Examples •  $\mathcal{U}$ : differential  $\mathbb{k}$ -vect. spaces,  $E = \tau = 0$  "trivial 2-repr."

•  $\mathcal{U} = \mathcal{U}, E = e \otimes -$

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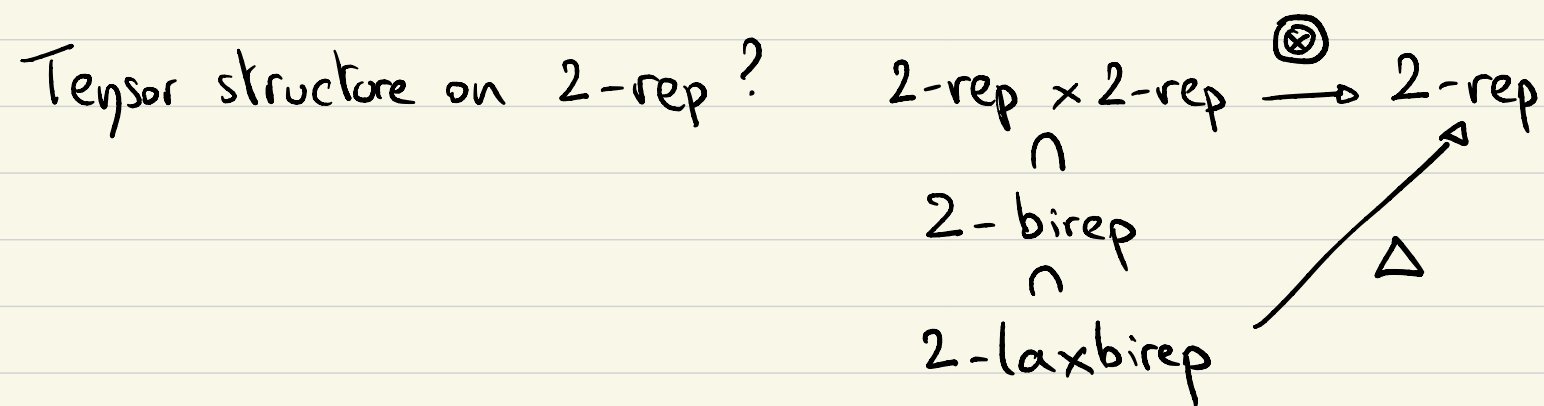
•  $\mathcal{U} = \mathcal{U}$  is a bimodule 2-repr: it is acted on by  $\mathcal{U} \otimes \mathcal{U}$

Lax bimod 2-rep on  $\mathcal{U}$ :  $(E_1, \tau_1), (E_2, \tau_2)$  2-rep and  $\sigma: E_2 E_1 \rightarrow E_1 E_2$  such that

$$\begin{array}{ccc} E_2^2 E_1 \xrightarrow{E_2 \sigma} E_2 E_1 E_2 \xrightarrow{\sigma E_2} E_1 E_2^2 & & \\ \tau_1 E_1 \downarrow & \curvearrowright & \downarrow E_1 \tau_1 \\ E_2^2 E_1 \xrightarrow{E_2 \sigma} E_2 E_1 E_2 \xrightarrow{\sigma E_2} E_1 E_2^2 & & \end{array}$$

$$\begin{array}{ccc} E_2 E_1^2 \xrightarrow{\sigma E_1} E_1 E_2 E_1 \xrightarrow{E_1 \sigma} E_1^2 E_2 & & \\ E_2 \tau_1 \downarrow & \curvearrowright & \downarrow \tau_1 E_2 \\ E_2 E_1^2 \xrightarrow{\sigma E_1} E_1 E_2 E_1 \xrightarrow{E_1 \sigma} E_1^2 E_2 & & \end{array}$$

# Coproduct



$(W, E_1, \tau_1, E_2, \tau_2, \sigma)$  a lax bimod 2-rep.

\* Define  $\Delta W$  a differential cat.

objects : pairs  $(m, \pi)$ ,  $m \in$  idempotent completion of pretriangulated closure of  $W$

$$\pi \in \text{Hom}(E_2(m), E_1(m)), \quad d(\pi) = 0$$

such that

$$\begin{array}{ccccccc}
 E_2^2(m) & \xrightarrow{E_2 \pi} & E_2 E_1(m) & \xrightarrow{\sigma} & E_1 E_2(m) & \xrightarrow{E_1 \pi} & E_1^2(m) \\
 \tau_2 \downarrow & & & \mathcal{D} & & & \downarrow \tau_1 \\
 E_2^2(m) & \xrightarrow{E_2 \pi} & E_2 E_1(m) & \xrightarrow{\sigma} & E_1 E_2(m) & \xrightarrow{E_1 \pi} & E_1^2(m)
 \end{array}$$

$$\text{Hom}((m, \pi), (m', \pi')) = \left\{ f \in \text{Hom}(m, m') \mid \begin{array}{ccc} E_2(m) & \xrightarrow{\pi} & E_1(m) \\ E_2 f \downarrow & \circlearrowright & \downarrow E_1 f \\ E_2(m') & \xrightarrow{\pi'} & E_1(m') \end{array} \right\}$$

\* Define  $E : \Delta(\mathcal{W}) \rightarrow \Delta(\mathcal{W})$  by

$$E(m, \pi) = \left( \text{Cone}(E_2(m) \xrightarrow{\pi} E_1(m)), \begin{pmatrix} \sigma \circ E_2 \pi \circ \tau_2 & \sigma \\ 0 & \tau_1 \circ E_1 \pi \circ \sigma \end{pmatrix} \right)$$

