Higher representations and Heegaard-Floer theory I

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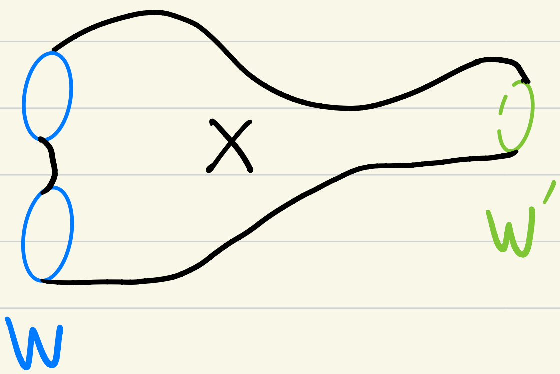
Topological quantum field theories

(d-1, d)-TQFT: W closed (compact smooth oriented) $(d-1)$-manifold $\mapsto Z(W)$ vect. sp/k (fin. dim.)

X compact d-manifold with $\partial X = -W \amalg W' \mapsto Z(X) : Z(W) \to Z(W')$



Require $Z(X)$ depends only on the cobordism class of X

Require • $Z(W_1 \amalg W_2) = Z(W_1) \otimes Z(W_2)$, $Z(\emptyset) = k$, $Z(W \times [0,1]) = id_{Z(W)}$

So: X closed $\rightsquigarrow Z(X) \in k$

Key: Can compute $Z(X)$ by cutting X in small pieces.

d=2: $A = Z(S^1)$. $Z(\quad) : A \otimes A \to A$, $Z(\quad) : A \to k$. A commutative Frobenius alg.

Extended 1,2,3 TQFT (Reshetikhin-Turaev, Bartlett-Douglas-Schommer-Pries-Vicary)

1-man $\mapsto$ category, 2-man $\mapsto$ functor, 3-man. $\mapsto$ natural transf.

$Z(S^1)$: modular category

Example: subquotient of $U_q(\mathfrak{g})$-mod, q a root of unity (WRT invariants)

Fully extended $0,1,\dots,n$ TQFT $\sim$ fully dualizable objects in (∞,n)-categ.

"Cobordism hypothesis" Galatius-Madsen-Tillmann-Weiss (invertible case)
Costello, Schommer-Pries, Hopkins ($d=2$)
(Baez-Dolan) Lurie, Ayala-Francis, Grady-Pavlov

Crane-Frenkel (1994): 1,2,3,4 TQFT from "categorified representation theory"?

1, ..., n TQFT's $\quad Z(S^1)$ monoidal $(n-2)$-category

$n=2$ forgetful functor is monoidal, commutes with $V\otimes W \simeq W\otimes V$

$n=3$ forgetful functor is monoidal, <u>does not</u> commute with $V\otimes W \simeq W\otimes V$
(R-matrix = braiding)

$n=4$ forgetful functor is not monoidal

"Hopf categories"

$\mathcal{C}$ monoidal cat $\rightsquigarrow$ $\mathcal{C}$-Mod, a 2-category

obj: $\mathcal{V}$ cat, mon. functor $\mathcal{C} \to \mathrm{End}(\mathcal{V})$

Hom: $\mathcal{H}om_{\mathcal{C}\text{-Mod}}(\mathcal{V},\mathcal{W}) = \{\phi : \mathcal{V} \to \mathcal{W}, (\alpha_c : c\phi \xrightarrow{\sim} \phi c)_{c \in \mathcal{C}} \mid \text{compatibilities}\}$

Want tensor product or internal Hom for $\mathcal{C}$-Mod. Underlying category?

$$\mathcal{H}om(\mathcal{V},\mathcal{W}) := \{\phi : \mathcal{V} \to \mathcal{W}, (\alpha_c : c\phi \to \phi c)_{c \in \mathcal{C}} \mid \text{compatibilities}\}$$

Hopf category = action of $\mathcal{C}$ on $\mathcal{H}om(\mathcal{C},\mathcal{C})$ (commuting with left and right $\mathcal{C}$-actions)

$\wr\wr$

compatible data of $\mathcal{C}$-action on all $\mathcal{H}om(\mathcal{V},\mathcal{W})$

Rem: duals by taking adjoints (if exist). Not an extra structure

Thm | $\mathcal{C}$ over a field, $\mathbb{Z}_{\geq 0}$-filtered and $gr(\mathcal{C})$ "nil-symmetric" monoidal cat.

$(c\otimes c' \to c'\otimes c \to c\otimes c'$ is $0)$

Then $\mathcal{C}$-Mod_{A_∞} admits a tensor structure.

Example : * $\mathfrak{g}$ symmetrizable Kac-Moody algebra.

2-representations of $\mathfrak{g}$ form a monoidal A_∞ 2-category

* Already for $\mathfrak{sl}_2^{>0}$: homotopical complications.

Vector rep of $\mathfrak{sl}_2$ $\longleftarrow\!\!\cdots\!\!\circ$ $\begin{pmatrix} k\text{-mod} \\ \oplus \\ k\text{-mod} \end{pmatrix} = \mathcal{L}(1)$ vector 2-rep of $\mathfrak{sl}_2$

Get $\mathcal{L}(1)^{\otimes n} \simeq \bigoplus_{i=0}^{n} D^b_{B\text{-smooth}}(Gr_i(n))$

* For $\mathfrak{gl}(1|1)$, homotopical complications in $\otimes$ disappear: rigid model (j. with A. Manion)