Notes available at bit.ly/KellersNotes

Lecture 3

Recall: k a field, $\mathcal{A}$ a (small) dg k-category, e.g. a dg algebra or ordinary algebra.

It has a Hochschild cochain complex $C(\mathcal{A},\mathcal{A})$ and $H^0C(\mathcal{A},\mathcal{A}) = HH^0(\mathcal{A}) = Z(\mathcal{A})$.

$C(A,A)$ carries a B_∞-structure and therefore $HH^*(\mathcal{A},\mathcal{A})$ carries a cup product Gerstenhaber bracket.

3.2 Singularity categories

A a right Noetherian k-algebra, e.g. $k[X_1, \ldots, X_n]/I$

$\operatorname{mod} A$ = cat. of finitely generated (right) A-modules

$\mathcal{D}^b(\operatorname{mod} A)$ = bounded derived category

$\cup\!\!\mathsf{I}$

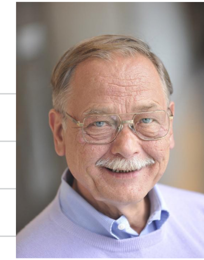
$\operatorname{per}(A)$ = perfect derived category = $\operatorname{thick}(A_A) \subseteq \mathcal{D}^b(\operatorname{mod} A)$

$sg(A) = \mathcal{D}^b(\mathrm{mod}\,A)/\mathrm{per}(A)$ (Verdier quotient)

$=$ stable derived cat. (Buchweitz 1986)

$=$ singularity cat. (Orlov 2003)

Jean-Louis Verdier 1935-1989



R.-O. Buchweitz 1952-2017

D. Orlov, 1966-

Rk: $sg(A) = 0$ if A is "smooth" (i.e. $\mathrm{gldim}\,A < \infty$)

Assume $A^e = A \otimes A^{op}$ is also Noetherian.

also called "Tate-Hochschild cohomology"

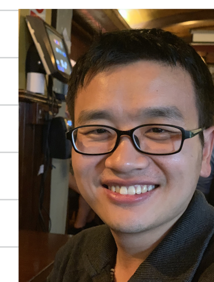
Def.: $HH^*_{sg}(A)$ = singular Hochschild cohom. $:= \mathrm{Ext}^*_{sg(A^e)}(A, A)$.

in any obvious way

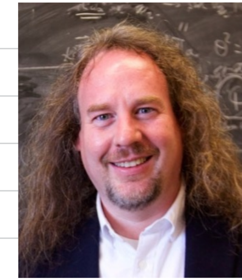
Rk: Not hard: $HH^*_{sg}(A)$ is still graded commutative. But $sg(A^e)$ is not monoidal.

Thm (Zhengfang Wang): a) $HH^*_{sg}(A)$ carries a natural (but very intricate!) Gerstenhaber bracket (2015).

b) There is a canonical B_∞-algebra $C_{sg}(A, A)$ computing $HH^*_{sg}(A)$ (2018).



Zhengfang Wang in '19



R. Kaufmann,1969-

Rk: Key tool for b): R. Kaufmann's spineless cacti operad (2007).

R. Kaufmann (2002)

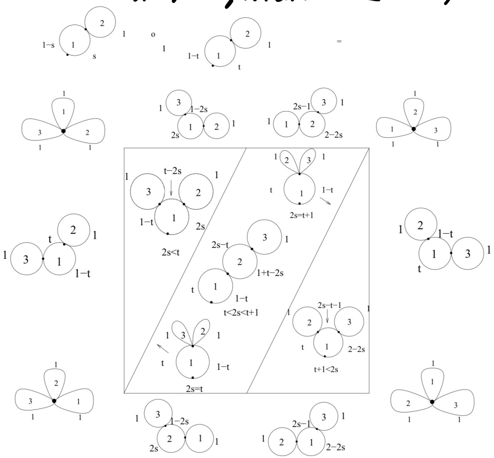


Figure 8: The associator in normalized spineless cacti

Z. Wang (2018)

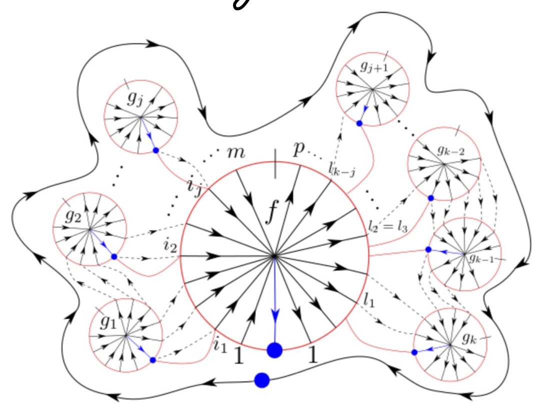


Figure 8: A summand $B^{(i_1,\cdots,i_j)}_{(l_1,\cdots,l_{k-j})}(f;g_1,\cdots,g_k)$ in the brace operation $f\{g_1,\cdots,g_k\}$.

Rk: So we have a complete structural analogy between singular and classical Hochsch. cohomology.

This suggests that HH^*_{sg} might be an instance of HH^*.

Main Thm: There is a canonical algebra morphism

can. dg enh. of $sg(A)$

$$\Psi: HH^*_{sg}(A) \longrightarrow HH^*(sg_{dg}(A))$$

which is "usually" invertible (e.g. if $\mathrm{char}\, k = 0$ and A is commutative).

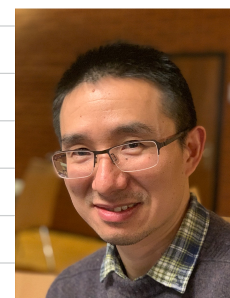
Rks: 1) It is not invertible if $k \subseteq A$ is a finite inseparable field extension. Then LHS $\neq 0$, RHS $= 0$.

2) The LHS is computable, the RHS is conceptually pleasing.

Conj: This (iso)morphism lifts to the B_∞-level.

Thm (Chen-Li-Wang): True for $A = kQ/(kQ_1)^2$, Q a finite quiver.

Example: Q: $\bullet \circlearrowright$ $\Rightarrow kQ/(kQ_1)^2 \simeq k[\varepsilon]/(\varepsilon^2)$.



Xiao-Wu Chen in 2019

Huanhuan Li in 2022

Isom. of the main thm: $\mathcal{M} = \mathcal{D}^b_{dg}(\mathrm{mod}\,A)$, $\mathcal{S} = \mathrm{sg}_{dg}(A)$.

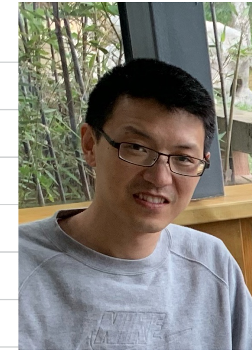
We have dg functors

$$A \xhookrightarrow{\ i\ } \mathcal{M} \xrightarrow{\ p\ } \mathcal{S}, \quad p \circ i \simeq 0$$

$$\begin{tikzcd}
\mathcal{D}^b(\mathrm{mod}\,A\otimes A^{op}) \arrow[r, "(\mathbb{1}\otimes i)^*"] \arrow[d] & \mathcal{D}(A\otimes\mathcal{M}^{op}) \arrow[r, "(i\otimes\mathbb{1})^!"] & \mathcal{D}(\mathcal{M}\otimes\mathcal{M}^{op}) \arrow[d, "(p\otimes p)^*"] \\
\mathrm{sg}(A\otimes A^{op}) \arrow[rr, dashed] & & \mathcal{D}(\mathcal{S}\otimes\mathcal{S}^{op}) \\
A \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[rr, mapsto, "\text{induces an isom. in } \mathrm{Ext}^*!"'] & & \mathcal{S} \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

✓

4.3 Application: reconstruction theorems for singularities, with Zheng Hua



Zheng Hua
in 2019

4.3.1 Hypersurface singularities

Thm 1 (Hua–K): $S = \mathbb{C}[\![x_1, \ldots, x_n]\!] \longrightarrow R = S/(f)$ isolated singularity.

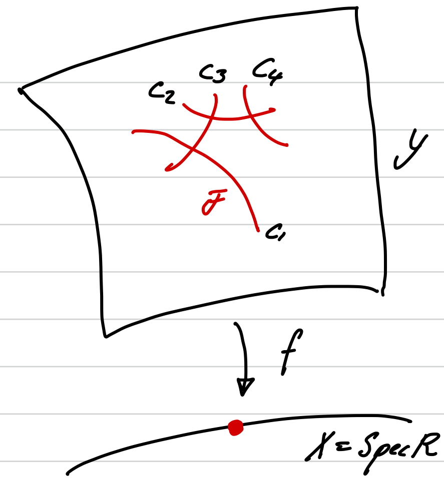
Then R is determined (up to isom.) by $\dim R$ and $sg_{dg}(R)$.

Sketch of proof:

$$\begin{tikzcd}
 & Z(sg_{dg}(R)) = HH^0(sg_{dg}(R)) \arrow[r, "\text{Main}\ \sim"', ""] \arrow[r, "\text{Thm}"'] & HH^0_{sg}(R) \arrow[d, "\wr"', "\text{Eisenbud '80}"] \arrow[d, "\text{matrix fact.}"'] \\
S/(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) = \text{Tyurina alg.} \arrow[r, "\sim", "\text{BACH } 1992"'] & HH^{2r}(R) \arrow[r, "\sim", "\text{Buchweitz '86}"'] & HH^{2r}_{sg}(R)
\end{tikzcd} \quad , \quad \forall r \gg 0$$

$\dim R$ and the Tyurina algebra determine R up to isomorphism (Mather–Yau 1982). $\checkmark$

4.3.2 Compound Du Val singularities



$k = \mathbb{C}$, R a complete local isolated cDV singularity

(3-dim., normal, generic hyperplane section is Du Val = Kleinian)

$f : \mathcal{Y} \to X = \operatorname{Spec} R$ a small crepant resolution (birational, isom.

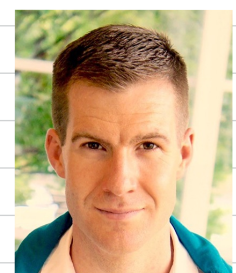
small $\Longleftrightarrow$ crepant

outside the exc. fiber, isom. in codim. 1, $f^*(\omega_X) = \omega_{\mathcal{Y}}$).

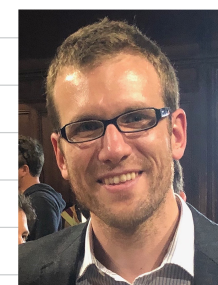
$\mathcal{F}$ = reduced exc. fibre : a tree of rat. curves $\mathcal{F} = \bigcup_{i=1}^{n} C_i$ contracted by f.

Associated (dg) algebras (cf. below) :

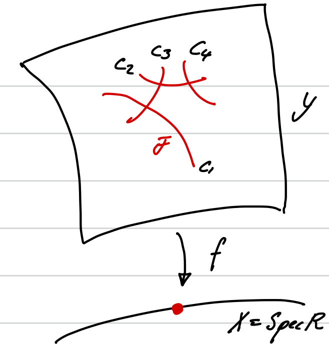
- contraction algebra Λ (Donovan-Wemyss, 2013).
- derived contraction algebra Γ



Will Donovan in '17



M. Wemyss in '20



$H^p\Gamma = 0,\ \forall p > 0$

Thm & def.: a) There is a can. connective dg algebra Γ which pro-represents the non com. deformations (non com. base, Laudal '02) of $\bigoplus_{i=1}^{n} \mathcal{O}_{C_i}$ in $\mathcal{D}^b(\mathrm{coh}\, Y)$ [Efimov-Lunts-Orlov 2010].

b) $H^0\Gamma$ is isomorphic [Hua–] to Λ which represents the non com. deformations of $\bigoplus_{i=1}^{n} \mathcal{O}_{C_i}$ in $\mathrm{coh}(Y)$ [Donovan-Wemyss, 2013]

Rks: 1) Λ is finite-dim. (like the Tyurina algebra) but non commutative.

Moreover, $H^p\Gamma$ is fin. dim. $\forall p \in \mathbb{Z}$.

Reid's width, bidegree of normal bundle

Katz' genus 0 Gopakumar-Vafa inv.

2) Λ determines many invariants of R (DW13, Toda '14, Hua-Toda '16, ...)

Conj. (DW '13): The derived equiv. class of Λ determines R itself (up to isomorphism).

smooth, left 3-CY

Thm 2 (Hua–): The derived eq. class of Γ determines R.

cluster category (Amiot '10)

H^*M of finite total dim.

Strategy: Show that

$$sg(R) \xrightarrow{\sim} \mathcal{C}_\Gamma := \mathrm{per}\,\Gamma / \mathcal{D}^{fd}\Gamma$$

even at the dg level, and use Thm 1 (Reid: R is a hypersurface). ✓

Rks: We have $H^*\Gamma \simeq \Lambda \otimes k[u^{-1}]$, $|u| = 2$, so Λ determines $H^*\Gamma$

but Γ is not formal! Nevertheless, there is hope because of the

following new approach.

inspired by G. Jasso's online minicourse
last week at Isfahan

http://portal.math.ipm.ir/PagesEvents/DisEventsHome.aspx?esbu=55eed02e-1bcc-4271-9c1e-73bde7da3dfb

Reformulation using cluster-tilting objects

Let $T \in \mathcal{C}_\Gamma \simeq sg(R)$ be the image of $\Gamma \in \mathrm{per}\,\Gamma$ under $\mathrm{per}\,\Gamma \to \mathcal{C}_\Gamma = \mathrm{per}\,\Gamma / \mathcal{D}^{fd}\Gamma$.

Amiot '09: T is a $2\mathbb{Z}$-cluster-tilting object of $\mathcal{C}_\Gamma$ [Iyama '07], i.e.

$$\mathrm{add}(T) = \{X \in \mathcal{C}_\Gamma \mid \mathrm{Ext}^i(T,X) = 0,\ \forall i \notin 2\mathbb{Z}\}$$

closure under finite direct sums and retracts

$$= \{X \in \mathcal{C}_\Gamma \mid \mathrm{Ext}^i(X,T) = 0,\ \forall i \notin 2\mathbb{Z}\}.$$

non derived.

Moreover, we have $\Lambda = H^0\Gamma \simeq \mathrm{End}(T)$.

The DW conjecture is implied by the more general:

CT-Conj: If $\mathcal{C}$ is a dg-enhanced triang. cat. (+ some technical hyp.) containing a $2\mathbb{Z}$-cluster-tilting object T, then $\mathcal{C}$ is determined by $\mathrm{End}_{\mathcal{C}}(T)$ (non derived!) up to quasi-equivalence of dg categories.

Rk: This means that the higher structure (the dg enhancement) is completely determined by lower structure (the non derived $\mathrm{End}_{\mathcal{C}}(T)$), which is surprising. We would get the DW Conj. as follows:

$$\Lambda = \begin{pmatrix}\text{contraction alg.} \\ \text{of a res. of } R\end{pmatrix} = \mathrm{End}_{\mathcal{C}_\Gamma}(T) \overset{\text{CT-conj.}}{\rightsquigarrow} (\mathcal{C}_\Gamma)_{dg} \simeq \mathcal{S}g_{dg}(R) \overset{\text{Thm 1}}{\rightsquigarrow} R \text{ up to isom.}$$

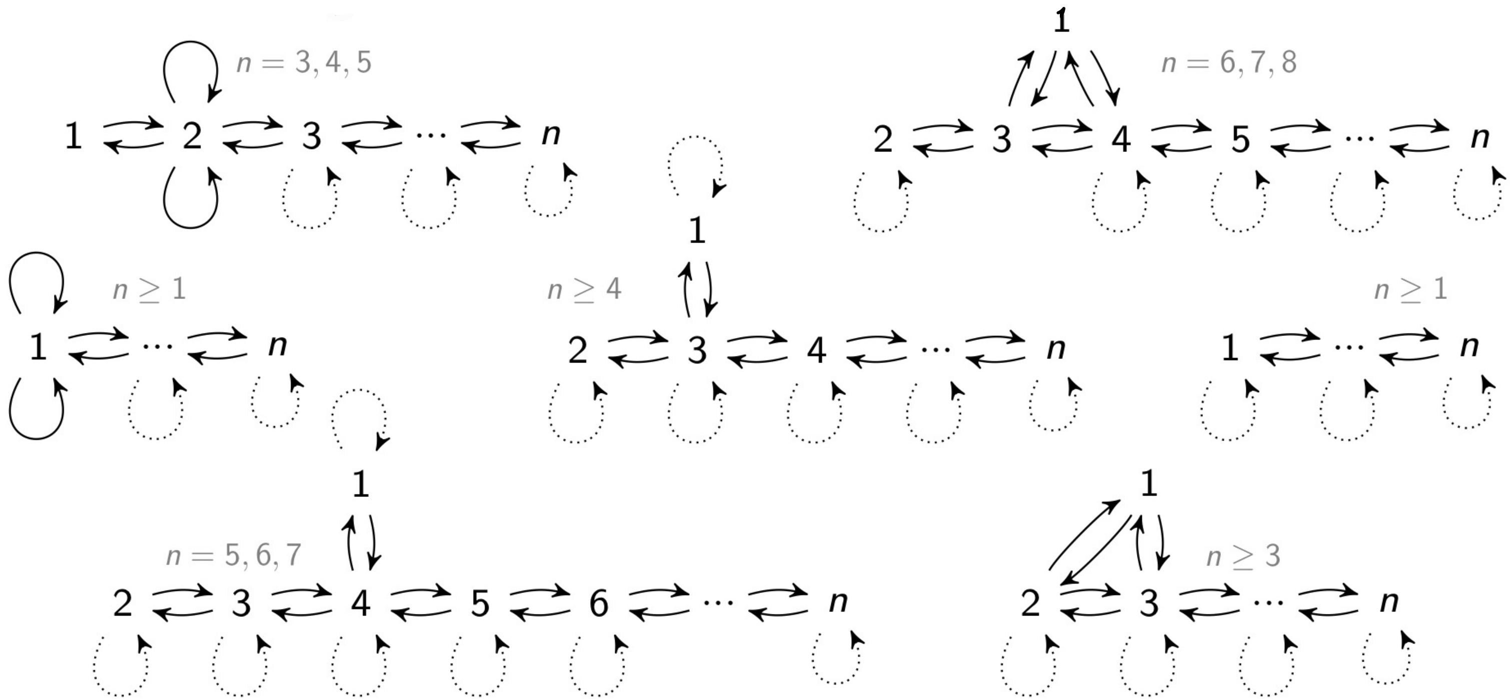
Rk: There is hope for the CT-Conj. because of the following 2 facts:

① Thm (Muro '22): Let $\mathcal{C}$ be a dg-enhanced triang. category (+hyp.) containing a $1\mathbb{Z}$-cluster-tilting object T. Then $\mathcal{C}$ is determined by $\mathrm{End}_{\mathcal{C}}(T)$ (non derived) up to quasi-equivalence of dg categories.

Examples of cat. with $1\mathbb{Z}$-cluster-tilting object: $\underline{CM}(R)$, where R is a simple singularity of even dimension.

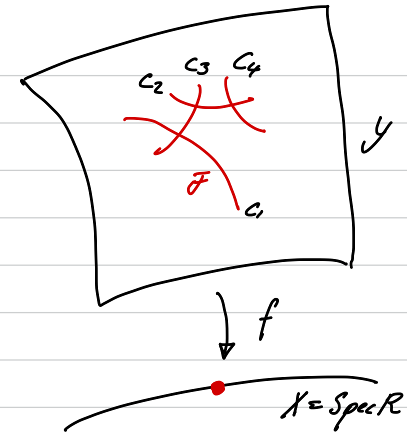
② Surprising feature of Iyama's "higher" homological theory: Many phenomena occurring in dimension 1 do generalize to higher dimensions!

Appendix A: Possible cDV quivers



Appendix B: Other constructions of Γ

B.1 Via tilting bundles (VdB '04)



Let $C_1, \dots, C_n$ be the irreducible components of the exc. fibre.

We have an isomorphism $\mathrm{Pic}(Y) \xrightarrow{\sim} \mathbb{Z}^n$, $\mathcal{L} \mapsto (\deg \mathcal{L}|_{C_i})_{1 \le i \le n}$.

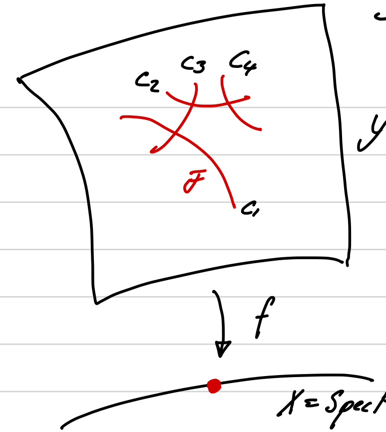
Let $\mathcal{L}_i$, $1 \le i \le n$, be line bundles s.th. $\deg(\mathcal{L}_i|_{C_j}) = \delta_{ij}$. Define $\mathcal{M}_i \in \mathrm{coh}(Y)$ as the "universal extension"

$$0 \longrightarrow \mathcal{O}_Y^{s_i} \longrightarrow \mathcal{M}_i \longrightarrow \mathcal{L}_i \longrightarrow 0$$

associated to a minimal set of generators of the R-module $H^1(Y, \mathcal{L}_i^{-1}) \cong \mathrm{Ext}^1(\mathcal{L}_i, \mathcal{O}_Y)$.

Then $\mathcal{T} = \mathcal{O}_Y \oplus \bigoplus_{i=1}^{n} \mathcal{M}_i$ is a tilting bundle on Y. Let $e \in \mathrm{End}(\mathcal{T})$ be the idempotent corresponding to the direct summand $\mathcal{O}_Y$ of $\mathcal{T}$ and $\tilde{\Gamma} = \mathrm{End}(\mathcal{T})$. Then $\Lambda = \tilde{\Gamma}/(e)$ and $\Gamma = \tilde{\Gamma} \overset{4}{/} (e)$.

The functor $f_*: \operatorname{coh} \mathbb{Y} \to \operatorname{mod} R$ sends $\mathcal{T}$ to a cluster-tilting Cohen-Macaulay module T (cf. B2) and induces an isomorphism $\operatorname{End}(\mathcal{T}) \xrightarrow{\sim} \operatorname{End}_R(T)$ so that this construction agrees with that of B.2.



B.2 Via Cohen-Macaulay modules

$\mathrm{cm}(R) = \{ M \in \mathrm{mod}\, R \mid \mathrm{Ext}^i_R(M,R) = 0,\ \forall i > 0 \}$.

Facts: $\mathrm{cm}(R)$ contains a cluster-tilting object T, i.e.

1) $\mathrm{Ext}^1(T,T) = 0$

2) $\forall M \in \mathrm{cm}(R)$, $\exists\ 0 \to T_1 \to T_0 \to M \to 0$, $T_i \in \mathrm{add}(T)$.

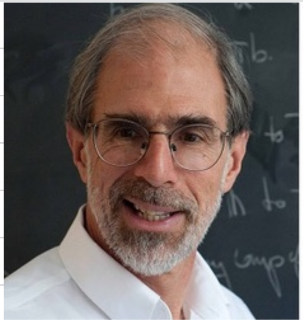
$\tilde{\Gamma} = \mathrm{End}_R(T)$ is independent of the choice of T up to derived equivalence.

We have $T = R^n \oplus T'$, where T' has no summands R. Let $e = \begin{bmatrix} \mathbb{1}_{R^n} & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{End}_R(T)$.

$\Rightarrow \Lambda \underset{\mathrm{der}}{\sim} \mathrm{End}_R(T)/(e)$ and $\Gamma \underset{\mathrm{der}}{\sim} \mathrm{End}_R(T) \overset{\mathbb{L}}{/} (e)$.

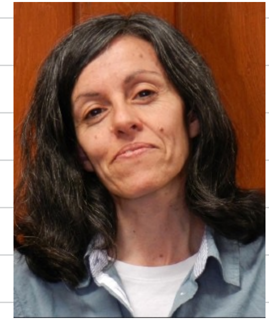
derived quotient.

B.3 Pictures : Mathematicians cited in the sketch of proof of Thm 1


David Eisenbud
1947-

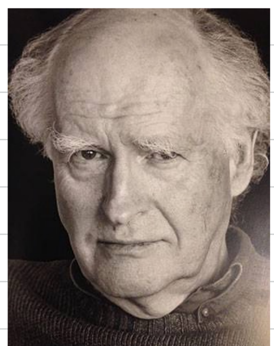

R.-O. Buchweitz
1952-2017


Orlando Villamayor
1923-1998


Andrea Solotar
in 2010


Galina Tyurina
1938-1970

John N. Mather
1942-2017

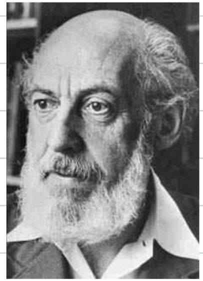


Stephen Yau
1952-

More mathematicians cited


G. Hochschild
1915-2010


H. Cartan
1904-2008


S. Eilenberg
1913-1998


M. Gerstenhaber, now 93

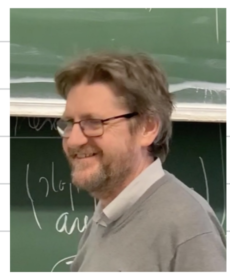

Ezra Getzler
1962-

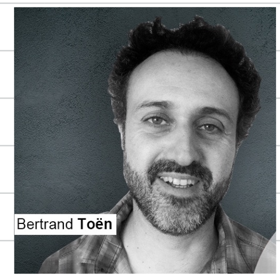

John D. S. Jones
1948-


T. Kadeishvili
1949-


B. Mitchell in 1981


W. Lowen in 2008


M. Van den Bergh
1960-



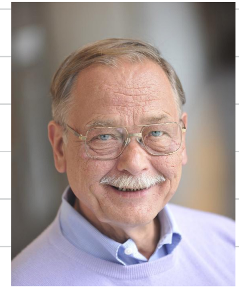
B. Toën, 1973-


V. Drinfeld, 1954-

H. Krause 1962-

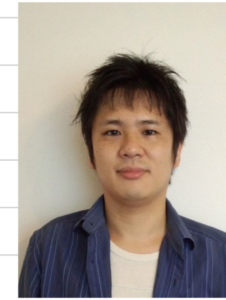
Yu Ye in 2019

Jean-Louis Verdier
1935-1989



R.-O. Buchweitz
1952-2017

D. Orlov, 1966-



Yukinobu Toda in ‘12

Will Donovan in ‘17

M. Wemyss in ‘20

C. Amiot in ‘08

S. Fomin, 1958-

A. Zelevinsky
1953-2013