Lecture 2

Notes available at bit.ly/KellersNotes

Reminder

k a field, A a k-algebra (assoc., with 1, non com.).

$a \longmapsto [a,?]$

$C(A,A)$ = Hochschild cochain complex $= (A \to \mathrm{Hom}_k(A,A) \to \mathrm{Hom}_k(A\otimes A, A) \to \dots)$

$C(A,A)$ carries: an A_∞-alg. structure, brace operations $m_{k,\ell}$, $k,\ell \geq 1$

augmented bar construction

$B^+(C(A,A))$ carries: a differential, a mult. $B^+C \otimes B^+C \longrightarrow B^+C$

$B^+(C(A,A))$ is a dg bialgebra $\Longleftrightarrow$: $C(A,A)$ is a B_∞-algebra,

Even a brace alg.: $Br = B_\infty / (m_{k,\ell},\ k>1)$.

Kontsevich-Soibelman '99: $k E_2 \simeq Br$ if char $k = 0$.

Next: Functoriality of the B_∞-structure on Hochschild cochains

Not.: $\mathcal{D}A$ = unbounded derived category of $\mathrm{Mod}\,A$ = {all right A-modules}

objects: all complexes $\cdots \to M^p \to M^{p+1} \to \cdots$ of right A-modules

morphisms: obtained from morphisms of complexes by formally inverting all quasi-isomorphisms $s: L \to M$ (i.e. $H^*s : H^*L \xrightarrow{\sim} H^*M$).

Thm ('03): Suppose that $X \in \mathcal{D}(A^{op} \otimes B)$ is such that $? \overset{L}{\underset{A}{\otimes}} X : \mathcal{D}A \to \mathcal{D}B$ is fully faithful. Then there is a canonical "restriction" morphism

$$res_X : C(B,B) \longrightarrow C(A,A)$$

in the homotopy category of B_∞-algebras (defined as the localization w.r.t. all q-i's of the cat. of B_∞-algebras). It is invertible if $X \overset{L}{\underset{B}{\otimes}} ? : \mathcal{D}(B^{op}) \to \mathcal{D}(A^{op})$ is also fully faithful.

Cor.: If $A = A_0 \oplus A_1 \oplus \dots$ is an (Adams-)graded Koszul algebra and

$$A^! = \bigoplus_{p,q} \mathrm{Ext}^p_A(A_0, A_0\langle q\rangle)$$ is (Adams-) graded Koszul dual,[1] we have

$$C(A,A) \simeq C(A^!, A^!)$$

in the homotopy category of (Adams-) graded B_∞-algebras.

Rk: Preservation of the cup product is due to Buchweitz

Idea of proof: Use $X = \bigoplus_{q\in\mathbb{Z}} R\mathrm{Hom}_A(A_0, A_0\langle q\rangle) \in \mathcal{D}^{\mathrm{Adams}}(A \otimes (A^!)^{op})$. ✓

[1] viewed as a dg algebra for the differential degree p with $d = 0$

Sketch of the construction of $res_X : C(B,B) \longrightarrow C(A,A)$ in the Thm:

dg category $\mathcal{G}$ with 2 objects

Let

$$G = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = \text{"glueing"} \quad \begin{tikzcd} \overset{B}{\bullet} \arrow[loop above] \arrow[r, "X"] & \overset{A}{\bullet} \arrow[loop above] \end{tikzcd} .$$

This is a dg (= differential graded) algebra. Let $C(G,G)$ be the product total complex of the Hochschild cochain complex of G. Let $R = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \subseteq \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$.

Let $C_R(G,G) \subseteq C(G,G)$ be the "R-relative" subcomplex given by

$$\mathrm{Hom}^{\bullet}_{R^e}(G^{\otimes_R p}, G) \subseteq \mathrm{Hom}^{\bullet}_k(G^{\otimes p}, G).$$

(Categorical interpretation: $C_R(G,G)$ is the H. cochain complex of the dg category $\mathcal{G}$ with 2 objects). The inclusion $C_R(G,G) \hookrightarrow C(G,G)$ is a qis of B_∞-algebras.

Idea: $C_R(G,G)$ is "intermediate" between $C(B,B)$ and $C(A,A)$:

$$\begin{tikzcd}
C_R(G,G) \arrow[r, "res_A"] \arrow[d, "res_B"', "\wr\ !"] & C(A,A) \\
C(B,B) \arrow[ur, dashed, bend right, "=: res_X"'] &
\end{tikzcd}$$

We have the diagram

$$\begin{tikzcd}
C_R(G,G) \arrow[r, "res_A"] \arrow[d, "res_B"', "\wr"] \arrow[dr, phantom, "\Leftarrow\ \square_h"] & C(A,A) \arrow[d, "\wr"] \arrow[dr, phantom, "\Leftarrow"] & RHom_{A^e}(A,A) \arrow[l, "\sim"'] \arrow[d, "\wr\ \text{action}"] \arrow[dr, phantom, "\Leftarrow"] & A \arrow[d, "\wr\ \text{faithfulness}"] \\
C(B,B) \arrow[r] & RHom_{A^{op}\otimes B}(X,X) & RHom_{A^e}(A, RHom_B(X,X)) \arrow[l, "\sim"'] & RHom_B(X,X)
\end{tikzcd}$$

Here res_A and res_B are morphisms of B_∞-algebras and res_B is a quasi-isomorphism.

We put $res_X = res_A \circ res_B^{-1}$. ✓

2. B_∞-algebras and monoidal categories (after Lowen-Van den Bergh and Lurie)

V a homologically unital B_∞-algebra.

$\mathrm{Mod}\, V = \{$ homolog. unital right A_∞-modules over $V \}$

$\mathcal{D}V$ = derived category $= (\mathrm{Mod}\, V)[\mathrm{qis}^{-1}]$

Lemma: $\mathcal{D}V$ "is" a monoidal triangulated cat. with unit $I = V$.

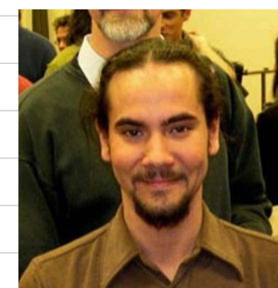
Proof (sketch):

$V^+ = V \oplus k$ = associated augmented A_∞-algebra, $C^+ = B^+ V = T^c(\Sigma V)$.

= conilpotent

$\mathrm{Com}(C^+) = \{$ complete right dg C^+-comodules $\}$

It becomes monoidal for $\otimes_k$ with unit k since C^+ is a dg bialgebra.



Kenji Lefèvre-Hasegawa in 2003

We have

$$\begin{tikzcd}
\mathcal{D}(V^+) \arrow[d, phantom, "\wr"] & \mathrm{Mod}\,V^+ \arrow[l] \arrow[d, shift left, "R"] & \mathrm{Mod}\,V \arrow[l, hook'] \arrow[r] \arrow[d, shift left, "R"] & (\mathrm{Mod}\,V)[\mathrm{qis}^{-1}] = \mathcal{D}V \arrow[d, phantom, "\wr"] \\
\mathcal{D}^{co}(C^+) & \mathrm{Com}\,C^+ \arrow[l] \arrow[u, shift left, "L"] & (\mathrm{Com}\,C^+)_{ac} \arrow[l, hook'] \arrow[r] \arrow[u, shift left, "L"] & (\mathrm{Com}\,C^+)_+[(R\,\mathrm{qis})^{-1}]
\end{tikzcd}$$

monoidal with unit V

coderived category

tensor ideal in $\mathrm{Com}\,C^+$

monoidal with unit RV

Here, we put $R = ? \otimes_\tau C^+$, $L = ? \otimes_\tau V^+$, $\tau : C^+ \to \Sigma V \simeq V \to V^+$ can. twisting cochain. ✓

Rk: It follows that

$$\mathrm{per}(V) = \text{perfect derived category} = \mathrm{thick}(V) \subseteq \mathcal{D}V$$

is also monoidal triangulated with unit V. closure under $\Sigma^{\pm 1}$, extensions, retracts

Thus, $\mathrm{per}(V)$ is a unitally generated monoidal triangulated category.

better: small, $\mathbb{E}_1$-monoidal, stable, k-lin. ∞-cat.

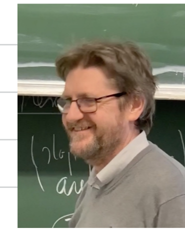
Philosophy: "Every" unitally gen. monoidal triang. cat. should be of this form!

Thm (Lowen–Van den Bergh, 2021): Let $(\mathcal{A}, \otimes, I)$ be a monoidal k-linear category s.th.

a) $\mathcal{A}$ is abelian (but $\otimes$ is not supposed exact!)

b) $\mathcal{A}$ has enough projectives and $? \otimes P$ is exact for projective P.

W. Lowen in 2008



M. Van den Bergh 1960-

Then $V = R\mathrm{End}_{\mathcal{A}}(I)$ carries a B_∞-structure s.th.

$$\text{can}: \begin{array}{ccc} \mathrm{per}\, V & \xrightarrow{\sim} & \mathrm{thick}(I) \subseteq \mathcal{D}\mathcal{A} \\ V & \longmapsto & I \end{array}$$

becomes monoidal.

Example: A an algebra, $\mathcal{A} = \mathrm{Mod}(A^e) = \{A\text{-bimodules}\}$, $\otimes = \otimes_A$, $I = {}_A A_A$.

Then $V = R\mathrm{Hom}_{A^e}(A, A) = C(A, A)$ as a dg algebra (up to qis)

and L-VdB show that their construction yields the classical B_∞-str. (up to qis).

Non example: X a top. space, $\mathcal{A} = Sh(X, Ab)$, $I = \underline{k}_X$.

Then $R\mathrm{End}(I) = C^*_{sg}(X, k)$ has Bauer's B_∞-str. but the Thm does not apply because $\mathcal{A}$ does not have enough projectives.

Jacob Lurie, 1977-

Lurie's thm [HA, Prop. 7.1.2.8]

Let R be an $\mathbb{E}_2$-ring spectrum.

Let $\mathcal{D}_\infty R$ be its ∞-enhanced derived category ($= \mathrm{Mod}_R$ in Lurie's not.).

It underlies an $\mathbb{E}_1$-monoidal stable ∞-cat. $(\mathcal{D}_\infty R)^\otimes$ ($= \mathrm{Mod}_R^\otimes$).

It is compactly generated by the tensor unit $I = R$.

Let $\mathrm{per}_\infty(R)^\otimes$ be its subcat. of compact objects. i.e. retracts of iterated extensions of shifts of R

$\mathrm{per}_\infty(R)^\otimes$ is a small $\mathbb{E}_1$-monoidal unitally generated stable ∞-category.

Thm (Lurie): The construction $R \mapsto \mathrm{per}_\infty(R)^\otimes$ yields an equivalence of ∞-cat.

$$\{\mathbb{E}_2\text{-ring spectra}\} \xrightarrow{\sim} \{\text{small } \mathbb{E}_1\text{-monoidal unitally gen. stable } \infty\text{-cat.}\}$$

Rk: Let k be a field of characteristic 0. Then the k-linearized $\mathbb{E}_2$-operad $k\mathbb{E}_2$ is quasi-isomorphic to the brace operad Br [KS99]. It seems very likely that we have the

Corollary in progress (Jasso): The construction $V \mapsto \mathrm{per}_{dg}(V)^\otimes$ yields an equiv. of ∞-cat.

$$\{Br_\infty\text{-algebras}\} \xrightarrow{\sim} \{\text{small } k\mathbb{E}_1\text{-monoidal unitally gen. stable dg cat.}\}$$

i.e. htpy Br-algebras

Rk: Recall that the brace operad Br is a quotient of the B_∞-operad B_∞:

$$Br = B_\infty / (m_{k,\ell},\ k > 1).$$

So each Br-alg. is also a B_∞-alg. The Corollary in progress yields the converse (!):

$$V \overset{?}{\longmapsto} V/(m_{k,l},\ k>1)$$

$$\begin{tikzcd}
V \arrow[d, mapsto] & \in & \{B_\infty\text{-algebras}\} \arrow[d] \arrow[rr, dashed, bend left] & & \\
(\mathrm{perf}_{dg} V)^{\otimes} & \in & \left\{\begin{array}{c}\text{small } \mathbb{E}_1\text{-mon.}\\ \text{unitally gen.}\\ \text{stable dg cat.}\end{array}\right\} & & \{Br_\infty\text{-alg.}\} \arrow[ll, "\sim"']
\end{tikzcd}$$

This suggests the picture

$$\begin{tikzcd}
B_\infty \arrow[d] \arrow[drr, bend left] & & \\
B_\infty/(m_{k,l},\ k>1) \arrow[rr, "\sim"] & & B_\infty/(m_{k,l},\ k>0) = Br
\end{tikzcd}$$

More on this at a future minicourse!