GAP XVII – Vancouver, May 16-20, 2022

B_∞-structures, monoidal categories and singularity categories

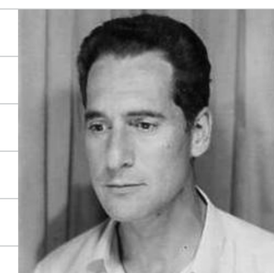
Plan:
1. B_∞-structures: From Hochschild to Getzler–Jones
2. Functoriality of the B_∞-structure on Hochschild cochains
3. B_∞-algebras and monoidal categories (after Lowen–Van den Bergh)
4. B_∞-structures for singularity categories

1. B_∞-structures: From Hochschild to Getzler–Jones

k a field, A a k-algebra (associative, with 1, non commutative)

$HH^*(A)$ = Hochschild cohomology of A (1945)

$= H^* C(A,A)$



G. Hochschild
1915-2010

$C(A,A)$ = Hochschild cochain complex

$$= \left(A \longrightarrow \mathrm{Hom}_k(A,A) \longrightarrow \mathrm{Hom}_k(A\otimes A, A) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_k(A^{\otimes p}, A) \longrightarrow \cdots\right)$$

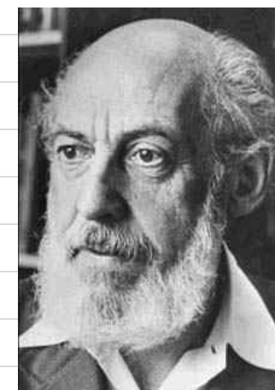
$$a \longmapsto (b \mapsto ab - ba),\quad D \longmapsto \big(a\otimes b \mapsto (Da)b - D(ab) + a\,D(b)\big)$$

We see:

$$HH^0(A) = \text{center of } A = \{a \in A \mid ab = ba,\ \forall b \in A\} = Z(A)$$: a com. alg.!

$$HH^1(A) = \mathrm{OutDer}_k(A)$$: a Lie algebra!

H. Cartan
1904-2008



S. Eilenberg
1913-1998

$A^e = A \otimes A^{op}$, $_A A_A$ = "identity bimodule"

Cartan–Eilenberg (1958):

$$HH^*(A) = \mathrm{Ext}^*_{A^e}(A,A),$$

an algebra for the cup product $\cup$.

Gerstenhaber (1963): • $HH^*(A)$ is graded commutative !

Modern argument: ${}_A A_A$ is the unit in $(\mathcal{D}(A^e), \overset{L}{\underset{A}{\otimes}})$.

• $HH^{*+1}(A)$ is a graded Lie algebra: Gerstenhaber bracket, which controls the deformations of A.

M. Gerstenhaber, now 94

Getzler-Jones (1994):

$(C(A,A), \cup, \text{brace op.})$ is a B_∞-algebra.

Hans Joachim Baues 1943-2020

"B" in "B_∞" for Baues (1981):

$C^*_{sg}(X,\mathbb{Z})$ is a B_∞-algebra.

topological space

singular cochains

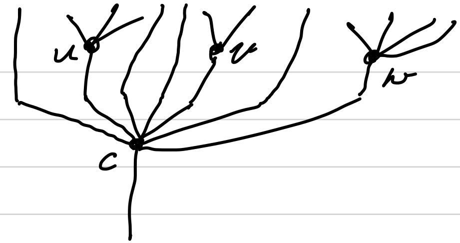
Mon. cat.: $(\mathcal{D}(Sh(X,Ab)), \overset{L}{\underset{\mathbb{Z}_X}{\otimes}})$

Ezra Getzler 1962-

John D. S. Jones 1948-

Brace operations (Kadeishvili 1988):

$$c\{u,v,\dots,w\} = \sum \pm$$




T. Kadeishvili, 1949-


Pierre Deligne, 1944-

Rk: 1) The B_∞-structure contains all the info, e.g. we have

$$[c,u] = c\{u\} \mp u\{c\}.$$

2) It is fundamental in (almost) all proofs of Deligne's conj.: $\mathbb{E}_2 \overset{\text{htpy}}{\circlearrowright} C(A,A)$.

Def. (Getzler-Jones '94): A B_∞-algebra is a $\mathbb{Z}$-graded vector space V together with a dg bialgebra structure $(T^c(\Sigma V), \Delta, \varepsilon, m, 1, d)$, where (Δ, ε) is the deconcatenation coalgebra structure on

$$B^+V = T^c(\Sigma V) = k \oplus \Sigma V \oplus \dots \oplus (\Sigma V)^{\otimes p} \oplus \dots$$

Rks: 1) Here B^+V is augmented (by def.) but V need not be.

2) The differential on $T^c(\Sigma V)$ yields an A_∞-algebra structure on V.

In the sequel, we often suppose it is homologically unital, i.e. H^*V is unital.

3) The B_∞-operad is a dg operad generated by operations μ_ℓ, $\ell \geq 2$, giving the A_∞-structure and by $m_{k,\ell}$ for $k, \ell \geq 0$ describing the multiplication $B^+V \otimes B^+V \longrightarrow B^+V$. The braces operad is the quotient by the operad ideal generated by the $m_{k,\ell}$, $k \geq 2$. It acts on $C(A,A)$ for any (A_∞-) algebra A. It is quasi-isomorphic to the E_2-operad in char. 0 (Kontsevich-Soibelman '99, cf. Willwacher '16, section 3).

2. Functoriality of the B_∞-structure on Hochschild cochains

Let A, B be k-algebras.

Rks: If $f: A \to B$ is an algebra morphism, it usually does not induce a morphism between the centers $Z(A) \dashrightarrow Z(B)$ and hence cannot induce a morphism in Hochschild cohomology. But we can gain some functoriality by passing to module categories: We have a can. isomorphism

$$\begin{array}{ccc} Z(A) & \xleftarrow{\sim} & Z(\mathrm{Mod}\,A) =: \mathrm{End}(\mathrm{Id}_{\mathrm{Mod}\,A}) \\ \varphi_A & \longleftarrow\!\shortmid & \varphi \end{array}$$

where $\mathrm{Mod}\,A = \{\text{all right } A\text{-modules}\}$ and $\mathrm{End}(\mathrm{Id}_{\mathrm{Mod}\,A})$ is the endomorphism algebra of the identity functor $\mathrm{Id}_{\mathrm{Mod}\,A}: \mathrm{Mod}\,A \to \mathrm{Mod}\,A$.

Thus, if $F : \mathrm{Mod}\,A \hookrightarrow \mathrm{Mod}\,B$ is any fully faithful functor, then we get a restriction morphism

$$\begin{tikzcd} Z(\mathrm{Mod}\,B) \arrow[r, "F^*"] \arrow[d, "\wr"'] & Z(\mathrm{Mod}\,A) \arrow[d, "\wr"] \\ Z(B) \arrow[r, dashed, "F^*"'] & Z(A) \end{tikzcd}, \quad (\varphi_M) \longmapsto (\psi_L) \text{ s.th.}$$

$$\begin{array}{ccc} \psi_L & \longmapsto & \varphi_{FL} \\ \cap & & \cap \\ \mathrm{End}(L) & \xrightarrow{\sim} & \mathrm{End}(FL) \end{array}$$

Aim: Construct a "derived analog" of $F^* : Z(B) \dashrightarrow Z(A)$, where the centers are replaced with Hochschild cochain complexes together with their B_∞-structure.