Poisson geometry of moduli of complexes

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Outline :

1) Shifted symplectic/Poisson structure

2) Moduli of complexes (on elliptic curves)

3) Bosonization (for Feigin-Odesskii mfd)

$k = \mathbb{C}$

§ p-forms and closed p-forms

$(C, d_C) \in \mathrm{cdga}_{\leq 0}$ assume $\mathbb{L}_C$ the cotangent complex of C is dualizable. Set $\mathbb{T}_C = \mathbb{L}_C^\vee$

<u>de Rham cplx</u> $\quad \overset{\text{weight}}{\Omega^\bullet_C} := \left(\mathrm{Sym}^\bullet(\mathbb{L}_C[-1]), \, d_{DR}, d_C \right)$

- d_{DR} has wt 1 deg 1, d_C has wt 0 deg 1
- $\Omega^\bullet_C$ is a "mixed graded object" in dg_k

For $n \in \mathbb{Z}$ the space of p-forms of deg n

$$A^p(C, n) := \left| \Omega^p_C[p+n] \right| \qquad \Omega^p_C = \mathrm{Sym}^p(\mathbb{L}_C[-1])$$

Recall if $E \in dg_k$ $|E|$ = the simplicial set associate to $\tau_{\leq 0} E$

the space of <u>closed</u> p-forms of deg n

$$A^{p\text{-}cl}(C, n) := \left| \hat{\Omega}^{\geq p}_C[p+n] \right|$$

$$\hat{\Omega}^{\geq p}_C = \left(\prod_{i \geq 0} \Omega^{p+i}[i] \, , \, d_{DR} + d_C \right)$$

Concretely a p-form of deg n is $(\alpha_0, \alpha_1, \dots)$

$$\begin{tikzcd}
\alpha_0 \in A^p(C,n) \arrow[r, "d_{DR}"] & A^{p+1}(C,n+1) & & \\
& \alpha_1 \in A^{p+1}(C,n) \arrow[u, "d_C"] \arrow[r] & A^{p+2}(C,n+1) & \\
& & \alpha_2 \in A^{p+2}(C,n) \arrow[u] \arrow[r] & \ddots
\end{tikzcd}$$

$$d_{DR}\alpha_0 = d_C \alpha_1$$

$$d_{DR}\alpha_1 = d_C \alpha_2$$

$$\vdots$$

$$A^{p,cl}(C,n) \longrightarrow A^p(C,n)$$

$$(\alpha_0, \alpha_1, \dots) \longmapsto \alpha_0$$

Def'n A n-shifted symplectic structure is $\omega = (\omega_0, \omega_1, \cdots) \in \mathcal{A}^{2,cl}(C, n)$ s.t.

$$\theta(\omega_0) : \mathbb{T}_C \longrightarrow \mathbb{L}_C[n] \text{ is a quasi-isom.}$$

Def'n Let $f : C \to D$ be a morphism in $cdga_{\leq 0}$.
Let ω be a n-shifted symp. structure on C.
f is called isotropic if $\exists$ homotopy

$$h : f(\omega) \sim 0 \quad \text{in} \quad \mathcal{A}^{2,cl}(D, n).$$

(f, ω, h) is called Lagrangian if

$$\begin{tikzcd} \mathbb{T}_D \arrow[r] \arrow[d] & \mathbb{T}_C \otimes D \arrow[r, "f(\omega_0)"] & \mathbb{L}_C \otimes D[n] \arrow[d] \\ 0 \arrow[rr] & & \mathbb{L}_D[n] \end{tikzcd}$$

is a homotopy fiber seq (exact)

We get (from exactness)

$$\Pi_{f,\omega,h} : \mathbb{L}_D[n] \longrightarrow \mathbb{T}_D[1]$$

is a bivector on D of deg $1-n$

Rmk 1) These definitions can generalized to derived stacks

2) Let $(\mathcal{X}, \omega_{\mathcal{X}})$, $(\mathcal{Y}, \omega_{\mathcal{Y}})$ be two stacks with n-shifted sympl. structures.

$f : \mathcal{L} \longrightarrow \mathcal{X} \times \mathcal{Y}$ is called a Lagrangian correspondence if $(f, (\omega_{\mathcal{X}}, -\omega_{\mathcal{Y}}), h)$ is Lagrangian.

Thm / Def'n (Melani-Safronov)

Given a n-shifted symplectic stack $(\mathcal{X}, \omega_{\mathcal{X}})$ and a Lagrangian structure $f : \mathcal{Y} \to \mathcal{X}$, $f^*\omega \overset{h}{\sim} 0$. $\Pi_{f,\omega,h} : \mathbb{L}_{\mathcal{Y}} \to \mathbb{T}_{\mathcal{Y}}[1-n]$ lifts to a $(n-1)$ shifted Poisson structure.

Rmk 1) if $\mathcal{X}$ is smooth, scheme then 0-shifted sympl./poisson str $\rightsquigarrow$ ordinary sympl./poisson structure.

2) if $t_0(\mathcal{X}) \longrightarrow X$ is a $\mathbb{G}_m$-gerbe over smooth coarse moduli then 2-forms / bivector descends to a 2-form / bivector on X.

§ Moduli of complexes

X : projective k-scheme

<u>Perf</u> (X) : (derived) stack of perfect $\mathcal{O}_X$-modules

Obj : bounded complex of vector bundles
equiv : quasi-isom

$\underline{\mathrm{Perf}}^{gr}(X)$: Stack of $\mathbb{Z}$-graded objects in $\mathrm{Perf}(X)$

$\varepsilon\underline{\mathrm{Perf}}(X)$: Stack of <u>mixed graded objects</u> in $\mathrm{Perf}(X)$

Obj: $\cdots \to V_0 \xrightarrow{\varepsilon} V_1 \xrightarrow{\varepsilon} V_2 \to \cdots$ boundar

$V_i \in \mathrm{Perf}(X)$ $\quad \varepsilon$: Morphism in $\mathrm{Perf}(X)$ $\quad \varepsilon^2 = 0$

Morphism = Morphism of $\mathrm{Perf}(X)$ commutes with ε

equiv = equiv in $\mathrm{Perf}^{gr}(X)$

Rmk 1) Suppose V_i are vector bundles

$$\begin{tikzcd} V_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"'] & V_1 \arrow[r] \arrow[d, "f_1"'] & \cdots \\ W_0 \arrow[r, "\varepsilon"] & W_1 \arrow[r] & \cdots \end{tikzcd}$$

is an equiv if f_i are isomorphisms

$[f, \varepsilon] = 0$

2) $X = \{pt\}$

$$\varepsilon \underline{\mathrm{Perf}}(\cdot) = \mathrm{Map}_{st}\left(\left[\mathbb{A}^1 / \mathbb{G}_m\right], \underline{\mathrm{Perf}}(\cdot)\right)$$

This is algebraic by the work of

[Halpern-Leistner Preygel]

<u>Thm</u> (Pantev-Toën-Vaquié-Vezzosi)

Let X be a smooth proj. CY d-fold with $\beta : \mathcal{O}_X \cong \Omega^d_X$

$\underline{\mathrm{Perf}}(X)$ admits a canonical $(2-d)$-shifted sympl. structure ω_β.

has a 2-shifted sympl. str. ω

Recall: $\underline{\mathrm{Perf}}(X) = \mathrm{Map}_{st}(X, \underline{\mathrm{Perf}}(\circ))$

$$\omega_\beta = \int_{[\beta]} ev^* \omega$$

$$
\begin{tikzcd}
 & \Sigma\underline{\mathrm{Perf}}(X) \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\underline{\mathrm{Perf}}^{gr}(X) & & \underline{\mathrm{Perf}}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{ V_0 \xrightarrow{\varepsilon} V_1 \to \cdots \} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\bigoplus V_i & & (\mathrm{Tot}_{\varepsilon + d})
\end{tikzcd}
$$

<u>**Thm A**</u> (H-Polishchuk)

X : dCY $\quad (p, q)$ is a Lagrangian correspondence w.r.t. ω_β. In particular $\Sigma\underline{\mathrm{Perf}}(X)$ admits a $(1-d)$ shifted Poisson structure.

<u>**Rmk**</u>

$$
\begin{tikzcd}
 & {[\mathbb{A}^1/\mathbb{G}_m]} & \\
B\mathbb{G}_m \arrow[ur, "i"] & & \mathbb{G}_m/\mathbb{G}_m \arrow[ul, "j"']
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
 & \cdot \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\cdot & & \cdot
\end{tikzcd}
$$

§ When $d=1$ $\quad X/k$ elliptic curve. $\quad [\bullet/\mathrm{Aut}]$

Thm B (H-P) Let $[v] \in \underline{\mathrm{Perf}}(X)$ be residual gerbe.

$S \subset \mathbb{Z}$ $\qquad [v^S] \in \prod_{i \in S} \underline{\mathrm{Perf}}(X) =: \underline{\mathrm{Perf}}^S(X)$

Then 1)

$$\begin{tikzcd} \star \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma\underline{\mathrm{Perf}}(X) \arrow[d, "q"] \\ {[v]} \arrow[r] & \underline{\mathrm{Perf}}(X) \end{tikzcd} \qquad \begin{tikzcd} \star \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma\underline{\mathrm{Perf}}(X) \arrow[d, "p^S"] \\ {[v^S]} \arrow[r] & \underline{\mathrm{Perf}}^S(X) \end{tikzcd}$$

(0-shifted)

are Poisson morphisms.

i.e. $\star$ are Poisson substacks of $\Sigma\underline{\mathrm{Perf}}(X)$.

2)
$$\begin{tikzcd} \star \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \varepsilon\underline{\mathrm{Perf}}(X) \arrow[d, "(p,q)"] \\ \left([V^{?}], [W]\right) \arrow[r] & \underline{\mathrm{Perf}}^{gr}(X) \times \underline{\mathrm{Perf}}(X) \end{tikzcd}$$

$\star$ is 0-symplectic.

__Examples__

1) Feigin-Odesskii mfd. ξ stable bundle on X of rk k < deg n

$$\mathcal{N}_\xi = \left\{ \mathcal{O} \to V_{n,k+1} \;\middle|\; V/\mathcal{O} \cong \xi \right\} \subset \varepsilon\underline{\mathrm{Perf}}(X)$$

$$\mathcal{N}_\xi \xrightarrow{} \mathbb{P}\mathrm{Ext}^1(\xi, \mathcal{O}_X)$$ is $\mathbb{G}_m$ gerbe

$N_\xi =$

$\rightsquigarrow$ FO bracket

Poisson bivector on $\mathcal{N}_3$ Fix $\{ \mathcal{O} \xrightarrow{d} V \} = v$

$$
\begin{tikzcd}
V^\vee \arrow[r, "\partial^*"] \arrow[d, "\partial^*"] & \mathcal{O} \oplus \operatorname{End} V \arrow[d, "0"] \\
\mathcal{O} \oplus \operatorname{End} V \arrow[r, "\partial"] & V
\end{tikzcd}
$$

$\partial^*(a) = ad + da$

$\partial(b, c) = db - cd$

$$T_v \mathcal{N}_3 = \mathbb{H}^\bullet(\mathcal{O} \oplus \operatorname{End} V \xrightarrow{\partial} V)[1]$$

$$L_v \mathcal{N}_3 = \mathbb{H}^\bullet(V^\vee \xrightarrow{\partial^*} \mathcal{O} \oplus \operatorname{End} V)$$

$X = \mathbb{C} / \mathbb{Z} + \mathbb{Z}\tau$

[FO] : N_3 is the semi-classical limits of elliptic algebras $Q_{n,k}(\eta, \tau)$

2) $K(X)$: field of meromorphic functions on X

$\mathrm{Mat}_{n\times n}(K(X))$ has a (multiplicative) (ind-) Poisson structure by the Belavin's elliptic r-matrix

$$\mathbb{P}\,\mathrm{Mat}_{n\times n}(K(X)) = \varinjlim_{D>0} \left\{ E \xrightarrow{\phi} E(D) \;\middle|\; \begin{array}{c} E \text{ stable} \\ \text{of rk } n \\ D^{>0} \text{ divisor} \end{array} \right\}$$

3) S : del Pezzo surface

X smooth element in $|-K_X|$

$S^{[n]}$ has a Poisson str π (Bottacin)

$S^{[n]}$ is the coarse moduli of certain components of $\mathcal{P}\mathrm{erf}(X)$ s.t the 0-shifted Poisson str descends to π.

<u>Example</u> $S = \mathbb{P}^2 \quad \mathcal{L} = \mathcal{O}(1)\big|_X$

$$\left\{ (\mathcal{L}^\vee)^{\oplus n} \xrightarrow{a} \mathcal{O}^{\oplus 2n+1} \xrightarrow{b} \mathcal{L}^{\oplus n} \;\middle|\; \begin{array}{l} ba = 0 \\ a \hookrightarrow \\ b \twoheadrightarrow \end{array} \right\}$$

$$\Big\downarrow \text{ stable quotient}$$

$$(\mathbb{P}^2)^{[n]}$$

§ Bosonization

Questions about $\mathcal{E}\underline{\mathrm{Perf}}(X)$

a) Sympl. leaves ⟵ $\boxed{\text{(Thm B.2)}}$

b) Symmetry (Poisson vector fields, bihamiltonian str ...)

c) Symplectic realization

d) Quantization

Bosonization.

Bosonization of $\mathcal{N}_\xi$

Recall $\mathcal{N}_\xi \simeq \{ \xi_{n,k} \xrightarrow{\phi} \mathcal{O}(1) \}$

$$= \{ \mathcal{O} \to V \mid V/\mathcal{O} \cong \xi \}$$

$\exists !$ seq.

$$k_0 = k > k_1 > k_2 > \cdots > k_p \geq 0$$

$$n_0 = n < n_1 < n_2 < \cdots < n_p$$

s.t. $k_i n_{i+1} - k_{i+1} n_i = 1$ & $(n_i, k_i) = 1$

Set $\vec{m} = (m_1, \cdots, m_p) \in \mathbb{N}^p$

A <u>bosonization of ϕ with multiplicity $\vec{m}$</u> is

a factorization

$$\xi \to \xi_1^{ss} \to \cdots \to \xi_p^{ss} \to \mathcal{O}(1)$$

with the composite being ϕ.

such that

i) $\operatorname{rk} \xi_i^{ss} = m_i \cdot k_i \qquad \deg \xi_i^{ss} = m_i n_i$

) ξ_i^{ss} are semi-stable

$\mathcal{B}(\xi, \vec{m}) :=$ moduli stack of all such factorizations

<u>Thm C</u> (HP) The composition map

$$\mu : \mathcal{B}(\xi, \vec{m}) \longrightarrow \mathcal{N}_\xi \text{ is Poisson}$$

↑ has a Poisson str.

Rmk $\mathcal{B}^{reg}(\xi, \vec{m}) = \left\{ \xi_i^{ss} \text{ are polystable with mutually non-isomorphic stable factors} \atop \text{morphism} \neq 0 \right\}$

$$f : \mathcal{B}^{reg}(\xi, \vec{m}) \longrightarrow \prod_{i=1}^{p} \mathcal{M}^{ss}_{n_i m_i, k_i m_i}$$

is a coisotropic fibration.

fiber : Rep variety of A_{p+2} quiver / torus

$\vec{m}_i >> 0$ μ is onto

<u>Application</u> $\vec{m} = (1, \cdots, 1)$

$B(\xi, \vec{m})$ has coarse moduli X^P

$\uparrow$ Poisson structure $= 0$

$\mu(X^P) \overset{\neq !!}{\subset}$ Zero locus of π on $N_\xi = \mathbb{P}\mathrm{Ext}^1(\xi, \mathcal{O})$

By analyzing $\mu(X^P)$ we obtain

1) (N_ξ, π) has zero Poisson vector field only.

2) π lifts uniquely to the linear space

$\mathrm{Ext}^1(\xi, \mathcal{O}_X)$