The ∞-groupoid of a pro-nilpotent L_∞-algebra and cubical sets

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Filtered complexes

We adopt the following shorthand: a filtered cochain complex $(L^\bullet, \delta, F)$ is a cochain complex (over a field $\mathbb{F}$ of characteristic zero, which we fix for the remainder of the talk) with a complete decreasing filtration

$$L = F^1 L \supset F^2 L \supset \cdots \supset 0, \quad F^p L = 0 \text{ for } p \gg 0,$$

such that $\delta(F^p L) \subset F^p L$. If the filtration is finite, we are in the nilpotent case.

Pro-nilpotent L_∞-algebras

A pro-nilpotent L_∞-algebra is a filtered cochain complex $(L^\bullet, \delta, F)$ with multilinear brackets $(k \geqslant 1)$

$$\{-,\dots,-\} : F^{p_1}L^{\ell_1} \times \cdots \times F^{p_k}L^{\ell_k} \to F^{p_1+\cdots+p_k+1}L^{\ell_1+\cdots+\ell_k+1},$$

satisfying two conditions:

- (symmetry) if $x_i \in L^{\ell_i}$, $1 \leqslant i < k$,
$$\{\dots, x_i, x_{i+1}, \dots\} = (-1)^{\ell_i \ell_{i+1}} \{\dots, x_{i+1}, x_i, \dots\};$$
- (generalized Jacobi)

$$\delta\{x_1,\dots,x_k\} + \sum_{i=1}^{k} \pm\{x_1,\dots,\delta x_i,\dots,x_k\}$$
$$+ \sum_{n=1}^{k} \sum_{\substack{i_1<\dots<i_n; j_1<\dots<j_{k-n} \\ \{i_1,\dots,i_n\}\cup\{j_1,\dots,j_{k-n}\}=\{1,\dots,k\}}} \pm\{\{x_{i_1},\dots,x_{i_n}\}, x_{j_1},\dots,x_{j_{k-n}}\} = 0.$$

The map $\delta x + \{x\}$ is a new differential on $L^\bullet$, deforming δ.

Let $\mathbf{g} = L[-1]$ be the suspension of L, with differential $dx = \delta x + \{x\}$ and bracket $[x, y] = (-1)^{|x|}\{x, y\}$. If the higher brackets $\{x_1, \dots, x_k\}$, $k > 2$, vanish, $\mathbf{g}$ is a nilpotent differential graded Lie algebra.

We may also consider curved L_∞-algebras, which have an additional operation $\{\}$ with $k = 0$: this is an element of F^1L^1, called the curvature. We do not discuss the curved case here, but most of our constructions extend to this case without difficulty.

For brevity, we refer to pro-nilpotent L_∞-algebras simply as L_∞-algebras. But it is important to not forget that our results only apply in the pro-nilpotent case, and not more generally.

Definition

The Maurer–Cartan locus $\mathrm{MC}(L)$ of an L_∞-algebra L is the set of solutions of the Maurer–Cartan equation on elements x of L of degree 0:

$$\delta x + \sum_{k=1}^{\infty} \frac{1}{k!} \underbrace{\{x, \dots, x\}}_{k \text{ factors}} = 0.$$

The Chevalley–Eilenberg complex of an L_∞-algebra

Given a filtered cochain complex $(L^\bullet, \delta, F)$, consider the (non-unital) differential graded cocommutative coalgebra

$$\mathsf{C}(L) = \prod_{n=1}^{\infty} \big(L^{\otimes n}\big)_{S_n}.$$

This is again a filtered complex.

A coderivation of $\mathsf{C}(L)$ is an operator D that satisfies the formula

$$(D \otimes 1 + 1 \otimes D)\nabla = \nabla D.$$

A coderivation is a codifferential if it has degree 1 and $D^2 = 0$.

There is a bijection between coderivations D and morphisms

$$\nu : \mathsf{C}(L) \to L.$$

We denote the coderivation associated to a morphism ν by the same symbol. This correspondence induces a bijection between codifferentials μ on $\mathsf{C}(L)$ of filtration degree 1 and (pro-nilpotent) L_∞-structures on L.

Given $x \in L^0$, denote by $\exp(x)$ the element

$$\exp(x) = \sum_{n=1}^{\infty} x^{\otimes n} \in \mathsf{C}(L).$$

This element is grouplike: it has degree 0 and satisfies the equation

$$\nabla \exp(x) = \exp(x) \otimes \exp(x).$$

Suppose the L_∞-structure on L is represented by the codifferential μ. The Maurer-Cartan equation for x is equivalent to the equation

$$(\delta + \mu) \exp(x) = 0.$$

Morphisms of L_∞-algebras

A morphism of L_∞-algebras $\mathbf{f} : L \to M$ is a morphism of filtered dg coalgebras $\mathbf{f} : \mathsf{C}(L) \to \mathsf{C}(M)$, that is, a morphism of filtered complexes satisfying

$$(\mathbf{f} \otimes \mathbf{f})\nabla = \nabla \mathbf{f}.$$

A morphism is determined by a sequence of maps $(k \geqslant 1)$

$$f_k : F^{p_1} L^{\ell_1} \times \cdots \times F^{p_k} L^{\ell_k} \to F^{p_1+\cdots+p_k} M^{\ell_1+\cdots+\ell_k},$$

of degree 0, satisfying two conditions:

- (symmetry) if $x_i \in L^{\ell_i}$, $1 \leqslant i \leqslant k$,

$$f_k(\ldots, x_i, x_{i+1}, \ldots) = (-1)^{\ell_i \ell_{i+1}} f_k(\ldots, x_{i+1}, x_i, \ldots);$$

- $\delta f_k(x_1, \dots, x_k)$

$$+ \sum_{\sigma \in S_k} \sum_{\ell=1}^{k} \sum_{k_1+\dots+k_\ell=k} \pm \{f_{k_1}(x_{\sigma(1)}, \dots), \dots, f_{k_\ell}(\dots, x_{\sigma(k)})\}$$

$$= \sum_{i=1}^{k} \pm f_k(\dots, \delta x_i, \dots)$$

$$+ \sum_{\sigma \in S_k} \sum_{\ell=1}^{k} \pm f_{k-\ell+1}(\{x_{\sigma(1)}, \dots, x_{\sigma(\ell)}\}, x_{\sigma(\ell+1)}, \dots, x_{\sigma(k)})$$

A morphism **f** has an underlying linear map $f_1 : L \to M$, of degree 0; the higher maps f_k are homotopies that correct for f_1 not preserving the brackets exactly.

A morphism is strict if $f_k = 0$, $k > 1$. The strict morphisms define the strict subcategory of the category of L_∞-algebras.

Lemma

A morphism $\mathbf{f} : L \to M$ *of* L_∞*-algebras induces a function* $\mathrm{MC}(\mathbf{f}) : \mathrm{MC}(L) \to \mathrm{MC}(M)$ *by the formula*

$$\mathbf{f}(x) = \sum_{k=1}^{\infty} f_k(x, \dots, x).$$

Proof.

This is a consequence of the formula $\exp(\mathbf{f}(x)) = \mathbf{f} \exp(x)$.

Homological perturbation theory

A contraction is a pair of filtered cochain complexes (V, δ) and (W, d), morphisms of filtered complexes $f : V \to W$ and $g : W \to V$ and a map $h : V \to V[-1]$, compatible with the filtration, such that

$$gf + \delta h + h\delta = 1_V, \qquad fg = 1_W, \qquad h^2 = fh = hg = 0$$

A Maurer-Cartan element $\mu \in F^1 \operatorname{End}(V)$ gives rise to a new contraction, by the formulas $\delta_\mu = \delta + \mu$ and

$$h_\mu = \sum_{n=0}^{\infty} (-h\mu)^n h \qquad d_\mu = d + \sum_{n=0}^{\infty} f(-\mu h)^n \mu g$$

$$g_\mu = \sum_{n=0}^{\infty} (-h\mu)^n g \qquad f_\mu = \sum_{n=0}^{\infty} f(-\mu h)^n.$$

Suppose that we have a contraction (L, M, f, g, h). This gives rise (via the tensor trick) to a contraction $(\mathsf{C}(L), \mathsf{C}(M), \mathbf{f}, \mathbf{g}, \mathbf{h})$, where

$$\mathbf{f} = \bigoplus_{n=0}^{\infty} f^{\otimes n}, \qquad \mathbf{g} = \bigoplus_{n=0}^{\infty} g^{\otimes n},$$

and $\mathbf{h}$ is the symmetrization of the homotopy on the tensor coalgebra

$$\bigoplus_{n=1}^{\infty} \sum_{k=1}^{n} (gf)^{k-1} \otimes h \otimes 1^{n-k},$$

which is given by the explicit formula

$$\mathbf{h} = \bigoplus_{n=1}^{\infty} \frac{1}{n} \sum_{\varepsilon_1,\dots,\varepsilon_{n-1}\in\{0,1\}} \binom{n-1}{\varepsilon_1 + \cdots + \varepsilon_{n-1}}^{-1} \sum_{i=1}^{n} (gf)^{\varepsilon_1} \otimes \ldots \otimes (gf)^{\varepsilon_{i-1}} \otimes h \otimes (gf)^{\varepsilon_i} \otimes \ldots \otimes (gf)^{\varepsilon_{n-1}}.$$

Recall that a Maurer–Cartan element μ in $F^1 \operatorname{End}(\mathsf{C}(L))$ corresponds to an L_∞-structure on L if it is a coderivation of $\mathsf{C}(L)$. Running the machinery of homological perturbation theory produces a differential d_μ on $\mathsf{C}(M)$ and morphisms of complexes $\mathbf{f}_\mu : \mathsf{C}(M) \to \mathsf{C}(L)$ and $\mathbf{g}_\mu : \mathsf{C}(M) \to \mathsf{C}(L)$.

It was shown by Berglund that d_μ is a coderivation of $\mathsf{C}(M)$

$$(d_\mu \otimes 1 + 1 \otimes d_\mu)\nabla = \nabla d_\mu.$$

and that $\mathbf{f}_\mu$ and $\mathbf{g}_\mu$ are morphisms of dg coalgebras

$$(\mathbf{f}_\mu \otimes \mathbf{f}_\mu)\nabla = \nabla \mathbf{f}_\mu \qquad\qquad (\mathbf{g}_\mu \otimes \mathbf{g}_\mu)\nabla = \nabla \mathbf{g}_\mu$$

In other words, d_μ is the codifferential associated to an L_∞-structure on M, $\mathbf{f}_\mu$ is an L_∞-map from L to M, whose linear term is f, and $\mathbf{g}_\mu$ is an L_∞-map from L to M, whose linear term is g.

Berglund actually works with general Koszul operads, but we need only the case of Lie and L_∞-algebras. This simplifies the discussion enormously.

Berglund's calculation

Since it is not so well known, I review Berglund's proof. Our presentation has been influenced by work in progress of Bandiera, which we use with his permission. It is based on the following lemma of Berglund.

Lemma

$$(\mathbf{f}_\mu \otimes \mathbf{f}_\mu)\nabla\mathbf{h} = 0$$

Proof.

We have $\mathbf{f}_\mu = \mathbf{f} - \mathbf{f}_\mu \mu \mathbf{h}$, so that $(\mathbf{f}_\mu \otimes \mathbf{f}_\mu)\nabla\mathbf{h}$ equals

$$\big(\mathbf{f}\otimes\mathbf{f} - (\mathbf{f}_\mu\mu\otimes 1)(\mathbf{h}\otimes\mathbf{f}) - (1\otimes\mathbf{f}_\mu\mu)(\mathbf{f}\otimes\mathbf{h}) + (\mathbf{f}_\mu\mu\otimes\mathbf{f}_\mu\mu)(\mathbf{h}\otimes\mathbf{h})\big)\nabla\mathbf{h}.$$

Each of these terms vanishes: $(\mathbf{f}\otimes\mathbf{f})\nabla\mathbf{h} = \nabla\mathbf{fh} = 0$, while we may check that $(\mathbf{h}\otimes\mathbf{f})\nabla\mathbf{h} = 0$, $(\mathbf{f}\otimes\mathbf{h})\nabla\mathbf{h} = 0$, and $(\mathbf{h}\otimes\mathbf{h})\nabla\mathbf{h} = 0$ using the explicit formula for $\mathbf{h}$.

Corollary

$(\mathbf{f}_\mu \otimes \mathbf{f}_\mu)\nabla\mathbf{h}_\mu = 0$ *and* $(\mathbf{f}_\mu \otimes \mathbf{f}_\mu)\nabla\mathbf{g}_\mu = \nabla$

Theorem

The linear map $\mathbf{f}_\mu : \mathsf{C}(L) \to \mathsf{C}(M)$ *is a morphism:* $(\mathbf{f}_\mu \otimes \mathbf{f}_\mu)\nabla = \nabla \mathbf{f}_\mu$.

Proof.

Take the differential of $(\mathbf{f}_\mu \otimes \mathbf{f}_\mu)\nabla \mathbf{h}_\mu = 0$ on the left and right:

$$\begin{aligned} 0 &= (d_\mu \otimes 1 + 1 \otimes d_\mu)(\mathbf{f}_\mu \otimes \mathbf{f}_\mu)\nabla \mathbf{h}_\mu + (\mathbf{f}_\mu \otimes \mathbf{f}_\mu)\nabla \mathbf{h}_\mu \delta_\mu \\ &= (\mathbf{f}_\mu \otimes \mathbf{f}_\mu)(\delta_\mu \otimes 1 + 1 \otimes \delta_\mu)\nabla \mathbf{h}_\mu + (\mathbf{f}_\mu \otimes \mathbf{f}_\mu)\nabla \mathbf{h}_\mu \delta_\mu \\ &= (\mathbf{f}_\mu \otimes \mathbf{f}_\mu)\nabla(\delta_\mu \mathbf{h}_\mu + \mathbf{h}_\mu \delta_\mu) \\ &= (\mathbf{f}_\mu \otimes \mathbf{f}_\mu)\nabla - (\mathbf{f}_\mu \otimes \mathbf{f}_\mu)\nabla \mathbf{g}_\mu \mathbf{f}_\mu = (\mathbf{f}_\mu \otimes \mathbf{f}_\mu)\nabla - \nabla \mathbf{f}_\mu. \end{aligned}$$

Corollary

The differential d_μ is a coderivation of $\mathsf{C}(M)$.

Proof.

Since $\mathbf{f}_\mu \mathbf{g} = (\mathbf{f} - \mathbf{f}_\mu \mu \mathbf{h})\mathbf{g} = 1$, we see that

$$\begin{aligned} \nabla \mathbf{f}_\mu \mu \mathbf{g} &= (\mathbf{f}_\mu \otimes \mathbf{f}_\mu)(\mu \otimes 1 + 1 \otimes \mu)(\mathbf{g} \otimes \mathbf{g})\nabla \\ &= (\mathbf{f}_\mu \mu \mathbf{g} \otimes \mathbf{f}_\mu \mathbf{g} + \mathbf{f}_\mu \mathbf{g} \otimes \mathbf{f}_\mu \mu \mathbf{g})\nabla \\ &= (\mathbf{f}_\mu \mu \mathbf{g} \otimes 1 + 1 \otimes \mathbf{f}_\mu \mu \mathbf{g})\nabla. \end{aligned}$$

The result follows, since $d_\mu = d + \mathbf{f}_\mu \mu \mathbf{g}$.

We turn to the proof that $\mathbf{g}_\mu$ is a morphism of coalgebras. Let

$$\pi = 1 - \delta h$$

be projection on the sum of the images of $g : M \to L$ and $h : L \to L$. Let

$$\boldsymbol{\pi} : \mathsf{C}(L) \to \mathsf{C}(L)$$

be its extension to $\mathsf{C}(L)$, defined by the tensor trick.

Lemma

$$(\mathbf{h}_\mu \otimes 1)\nabla \mathbf{g}_\mu = (1 \otimes \mathbf{h}_\mu)\nabla \mathbf{g}_\mu = 0$$

Proof.

Using that $\boldsymbol{\pi}\mathbf{g} = \mathbf{g}$, $\boldsymbol{\pi}\mathbf{h} = \mathbf{h}$ and $\mathbf{h}\boldsymbol{\pi} = 0$, we see that $\boldsymbol{\pi}\mathbf{g}_\mu = \mathbf{g}_\mu$ and $\mathbf{h}_\mu\boldsymbol{\pi} = \mathbf{h}$. Since $\nabla\boldsymbol{\pi} = (\boldsymbol{\pi} \otimes \boldsymbol{\pi})\nabla$, the result follows.

Theorem

The linear map $\mathbf{g}_\mu : M \to M$ *is a morphism:* $(\mathbf{g}_\mu \otimes \mathbf{g}_\mu)\nabla = \nabla \mathbf{g}_\mu$.

Proof.

Take the differential of $\big(\mathbf{h}_\mu \otimes 1 + \mathbf{g}_\mu \mathbf{f}_\mu \otimes \mathbf{h}_\mu\big)\nabla \mathbf{g}_\mu = 0$ on the left and right:

$$\begin{aligned} 0 &= (\delta_\mu \otimes 1 + 1 \otimes \delta_\mu)(\mathbf{h}_\mu \otimes 1 + \mathbf{g}_\mu \mathbf{f}_\mu \otimes \mathbf{h}_\mu)\nabla \mathbf{g}_\mu \\ &\quad + (\mathbf{h}_\mu \otimes 1 + \mathbf{g}_\mu \mathbf{f}_\mu \otimes \mathbf{h}_\mu)\nabla \mathbf{g}_\mu d_\mu \\ &= \big((\delta_\mu \mathbf{h}_\mu + \mathbf{h}_\mu \delta_\mu) \otimes 1 + \mathbf{g}_\mu \mathbf{f}_\mu \otimes (\delta_\mu \mathbf{h}_\mu + \mathbf{h}_\mu \delta_\mu)\big)\nabla \mathbf{g}_\mu \\ &= \big((1 - \mathbf{g}_\mu \mathbf{f}_\mu) \otimes 1 + \mathbf{g}_\mu \mathbf{f}_\mu \otimes (1 - \mathbf{g}_\mu \mathbf{f}_\mu)\big)\nabla \mathbf{g}_\mu \\ &= (1 \otimes 1 - \mathbf{g}_\mu \mathbf{f}_\mu \otimes \mathbf{g}_\mu \mathbf{f}_\mu)\nabla \mathbf{g}_\mu \\ &= \nabla \mathbf{g}_\mu - (\mathbf{g}_\mu \otimes \mathbf{g}_\mu)(\mathbf{f}_\mu \otimes \mathbf{f}_\mu)\nabla \mathbf{g}_\mu \\ &= \nabla \mathbf{g}_\mu - (\mathbf{g}_\mu \otimes \mathbf{g}_\mu)\nabla. \end{aligned}$$

There are quite different formulas for d_μ and $\mathbf{g}_\mu$ predating the work of Berglund. These formulas are analogous to the formulas of Kadeishvili and their generalizations in the theory of A_∞-algebras. The analogous formulas for L_∞-algebras replace sums over rooted planar trees by sums over rooted trees. In these trees, the vertices and edges represent the brackets of L and the homotopy h respectively.

Dotsenko, Shadrin and Vallette have proved that these two distinct formulas for d_μ and $\mathbf{g}_\mu$ yield the same result. We will have no need for this surprising result, which is not straightforward to prove.

Homotopies as gauge conditions

Consider the subset $\mathrm{MC}(L, h) \subset \mathrm{MC}(L)$ consisting of Maurer-Cartan elements $x \in \mathrm{MC}(L)$ satisfying the gauge condition $hx = 0$.

The idea of imposing this condition on Maurer–Cartan elements goes back to Kuranishi. He considered the case where $L = A^{0,*}(X, TX)$ is the Dolbeault resolution of the tangent bundle TX on a compact complex Hermitian manifold X, and

$$h = \left(\bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*\right)^{-1}\bar{\partial}^*.$$

The condition $hx = 0$, or equivalently $\bar{\partial}^* x = 0$, is analogous to the Lorentz condition in Maxwell's theory of electromagnetism $\operatorname{div} A = 0$, where A is a connection 1-form on a complex line bundle.

Theorem

On restriction to $\mathrm{MC}(L, \mathbf{h}) \subset \mathrm{MC}(L)$*, the morphism* $\mathbf{f}_\mu : \mathrm{MC}(L) \to \mathrm{MC}(M)$ *induces a bijection* $f : \mathrm{MC}(L, h) \to \mathrm{MC}(M)$*, with inverse* $\mathbf{g}_\mu$*.*

Proof.

If $x \in \mathrm{MC}(L, h)$, we have $\mathbf{h}\exp(x) = 0$, hence

$$\exp(\mathbf{f}_\mu(x)) = \mathbf{f}_\mu \exp(x) = \mathbf{f}(1 + \mu\mathbf{h})^{-1} \exp(x) = \mathbf{f}\exp(x) \\ = \exp(f(x)).$$

Hence

$$\exp(\mathbf{g}_\mu \mathbf{f}_\mu(x)) = \mathbf{g}_\mu \mathbf{f}\exp(x) = (1 + \mathbf{h}\mu)^{-1}\mathbf{g}\mathbf{f}\exp(x) \\ = (1 + \mathbf{h}\mu)^{-1}(1 - \delta\mathbf{h} - \mathbf{h}\delta)\exp(x).$$

Since $(\delta + \mu)\exp(x) = 0$, it follows that

$$\exp(\mathbf{g}_\mu \mathbf{f}_\mu(x)) = (1 + \mathbf{h}\mu)^{-1}(1 + \mathbf{h}\mu)\exp(x) = \exp(x),$$

proving that $\mathbf{g}_\mu \mathbf{f}_\mu$ equals the identity on $\mathrm{MC}(L, h)$.
If $y \in \mathrm{MC}(M)$, then $\mathbf{h}\exp(\mathbf{g}_\mu(y)) = \mathbf{h}\mathbf{g}_\mu \exp(y) = 0 \in \mathrm{C}(L)$. Since

$$\mathbf{h}\mathbf{g}_\mu \exp(y) = h\mathbf{g}_\mu(y) + \dots,$$

we see that $\mathbf{g}_\mu : \mathrm{MC}(M) \to \mathrm{MC}(L, h) \subset \mathrm{MC}(L)$.

Thom–Sullivan differential forms

We now turn to an application of the above general constructions in (higher) Lie theory. The Lie theory of an L_∞-algebra L is the theory of flat superconnections on simplicial complexes.

The geometric n-simplex Δ^n is the convex set

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{\{0,\dots,n\}} \mid t_i \geqslant 0 \text{ and } t_0 + \cdots + t_n = 1\}.$$

The simplicial differential graded algebra $\Omega(\Delta^\bullet)$ takes $[n]$ to differential forms with polynomial coefficients on the n-simplex Δ^n:

$$\begin{aligned}&\Omega(\Delta^\bullet)\\ &\quad = \mathbb{F}[t_0, \dots, t_n, dt_0, \dots, dt_n]/(t_0 + \cdots + t_n = 1, dt_0 + \cdots + dt_n = 0).\end{aligned}$$

The tensor product $\Omega(\Delta^\bullet) \otimes L$ of an L_∞-algebra L with the simplicial differential graded commutative algebra $\Omega(\Delta^\bullet)$ is a simplicial L_∞-algebra.

Homotopy theory of L_∞-algebras

Definition

The nerve $\mathsf{MC}_\bullet(L)$ of an L_∞-algebra simplicial set is the Maurer–Cartan set of the simplicial L_∞-algebra $\Omega(\Delta^\bullet) \otimes L$:

$$\mathsf{MC}_\bullet(L) = \mathsf{MC}(\Omega(\Delta^\bullet) \otimes L).$$

The homotopy groups $\pi_i(L)$ of an L_∞-algebra L are the homotopy groups $\pi_i(\mathsf{MC}_\bullet(L))$ of its nerve.

The functor $\mathsf{MC}_\bullet(L)$ is related to Sullivan's functor $\langle A \rangle$ that takes a dg commutative algebra A to a simplicial set: there is a natural equivalence of functors

$$\mathsf{MC}_\bullet(L) \cong \langle \mathsf{C}(L)^\vee \rangle.$$

Elementary differential forms

Consider the subcomplex of $\Omega^\bullet(\Delta^n)$ spanned by Whitney's complex of elementary differential forms. This is the subcomplex $C^\bullet(\Delta^n) \subset \Omega^\bullet(\Delta^n)$ with basis

$$\omega_{i_0\dots i_k} = k! \sum_{j=0}^{k} (-1)^i t_{i_j} dt_{i_0} \dots \widehat{dt}_{i_j} \dots dt_{i_k}, \quad 0 \leqslant i_0 < \cdots < i_k \leqslant n.$$

Theorem (Whitney)

The complex $C^\bullet(\Delta^n)$ is naturally isomorphic to the complex of normalized simplicial cochains (with coefficients in $\mathbb{F}$) of the n-simplex.

Dupont reformulates the de Rham theorem by constructing a simplicial homotopy $s_\bullet : \Omega^i(\Delta^\bullet) \to \Omega^{i-1}(\Delta^\bullet)$ such that

$$ds_n + s_n d = 1 - p_n,$$

where p_n is a projection onto the subcomplex $C^\bullet(\Delta^n) \subset \Omega^\bullet(\Delta^n)$.

Applying homological pertubation theory to the simplicial contraction $p_\bullet : \Omega^\bullet(\Delta^\bullet) \otimes L \to C^\bullet(\Delta^\bullet) \otimes L$ of cochain comlexes induced by the Dupont homotopy, we obtain L_∞-morphisms

$$C^\bullet(\Delta^\bullet) \otimes L \xrightarrow{\mathbf{i}_\bullet} \Omega^\bullet(\Delta^\bullet) \otimes L \xrightarrow{\mathbf{p}_\bullet} C^\bullet(\Delta^\bullet) \otimes L$$

whose composition is the identity map.

The simplicial set

$$\begin{aligned} \gamma_\bullet(L) &= \mathrm{MC}\big(\Omega(\Delta^\bullet) \otimes L, s_\bullet\big) \\ &= \big\{x \in \mathrm{MC}(\Omega(\Delta^\bullet) \otimes L) \bigm| s_\bullet x = 0\big\} \end{aligned}$$

is isomorphic to $\mathrm{MC}_\bullet(C^\bullet(\Delta^\bullet) \otimes L)$, and we have morphisms of simplicial sets

$$\gamma_\bullet(L) \subset \mathrm{MC}_\bullet(L) \xrightarrow{\mathbf{p}_\bullet} \gamma_\bullet(L)$$

Theorem

1. *The inclusion $\gamma_\bullet(L) \subset \mathrm{MC}_\bullet(L)$ is a homotopy equivalence.*
2. *If $L^i = 0$ for $i < -k$, the simplicial set $\gamma_\bullet(L)$ is a k-groupoid.*

The functor $\gamma_\bullet(L)$ specializes to well-known functors in two cases.

- If L is concentrated in degree -1, its suspension is a nilpotent Lie algebra, and $\gamma_\bullet(L)$ is isomorphic to the nerve of the associated Lie group G.
- If L is an abelian differential graded Lie algebra (i.e. cochain complex), then $\gamma_\bullet(L)$ is naturally isomorphic to the Eilenberg-Maclane space $K(\tau^{\leqslant 0}L)$ associated to the chain complex

$$(\tau^{\leqslant 0}L)^i = \begin{cases} L^i, & i < 0, \\ Z^0(L), & i = 0, \\ 0, & i > 0. \end{cases}$$

The functor $\gamma_\bullet(L)$ generalizes the Baker–Campbell–Hausdorff Theorem to differential graded Lie algebras, and indeed to (curved) L_∞-algebras.

More on Dupont's homotopy

The vector field

$$E_i = \sum_{j=0}^{n} (t_j - \delta_{ij}) \mathbf{p}_j$$

generates dilation centered at the vertex e_i of the n-simplex. Let $\phi_i(u)$ be the flow that it generates. The Poincaré homotopy is

$$h_n^i = \int_0^1 u^{-1}\, \phi_i(u)\, \iota(E_i)\, du.$$

Let $\varepsilon_n^i : \Omega(\Delta^n) \to \mathbb{F}$ be evaluation at the vertex e_i. Stokes's theorem implies the Poincaré lemma.

Lemma

h_n^i is a chain homotopy between the identity and ε_n^i:

$$dh_n^i + h_n^i d = \mathrm{Id}_n - \varepsilon_n^i.$$

Dupont's homotopy is given by the formula

$$s_n = \sum_{k=0}^{n-1} \sum_{0 \leqslant i_0,\dots,i_k \leqslant n} t_{i_0} dt_{i_1} \dots dt_{i_k} h_n^{i_k} \dots h_n^{i_0} .$$

This operator becomes increasingly complicated as n increases. We can however make some general statements:

- the kernel of s_n on $\Omega^0(\Delta^n)$ is of course the whole subspace;
- the kernel on $\Omega^k(\Delta^n)$ decreases in size as the degree k increases;
- the kernel on $\Omega^n(\Delta^n)$ is the rank 1 subspace spanned by

$$dt_1 \wedge \dots \wedge dt_n;$$

- the operator s_n is equivariant under the action of the permutation group S_{n+1} on the vertices of Δ^n.

In particular, s_1 is equivariant under the reversal map $(t_0, t_1) \mapsto (t_1, t_0)$, which exchanges the two boundary points of the 1-simplex.

Dupont's homotopy on the 1-simplex Δ^1

Dupont's homotopy is far simpler to understand when $n = 1$ than in the general case. In terms of the coordinate $t = t_1 = 1 - t_0$ on Δ^1, we have

$$\begin{aligned} s_1 a(t)\,dt &= t_0 h^0 a(t)\,dt + t^1 h^1 a(t)\,dt \\ &= (1-t)\int_0^t a(s)\,ds + t\left(-\int_t^1 a(s)\,ds\right) \\ &= \int_0^t a(s)\,ds - t\int_0^1 a(s)\,ds. \end{aligned}$$

This formula confirms that the kernel of s_1 on 1-forms on the 1-simplex is the span of dt.

Elements of $\mathrm{MC}_1(L)$ are the superconnections

$$x(t) + a(t)dt,$$

where $x(t)$ and $a(t)$ are in $L^1[t]$ and $L^0[t]$ respectively, satisfying the equations

$$\begin{cases} \delta x(t) + \frac{1}{2}[x(t), x(t)] = 0, \\ \dot{x}(t) + [a(t), x(t)] + \delta a(t) = 0. \end{cases}$$

The gauge condition $s_1 a(t)dt = 0$ imposes the additional equation $\dot{a}(t) = 0$, in other words, a is constant.

If L is a Lie algebra, $x(t) = 0$ and $\gamma_1(L)$ may be identified with the underlying set of the vector space L^1.

$\gamma_2(L)$ is the multiplication table of $\gamma_1(L)$.

Cubical nerves of differential graded Lie algebras

We will work with cubical sets with upper, or both lower and upper, connections. Our constructions are compatible with cubical sets with reversals, but not with exchange or diagonal.

There is a cubical space $\square^\bullet$, where $\square^n = [0,1]^n$. Denote points of $\square^n$ by $(x_1, \dots, x_n)$. The cofaces are the maps

$$\mathbf{p}^i_\varepsilon : \square^{n-1} \to \square^n, \quad i \in \{1, \dots, n\}, \quad \varepsilon \in \{0, 1\},$$

taking $(x_1, \dots, x_{n-1})$ to $(x_1, \dots, x_{i-1}, \varepsilon, x_i, \dots, x_n)$.

The codegeneracies are the maps

$$\sigma^i : \square^n \to \square^{n-1}, \quad i \in \{1, \dots, n\},$$

taking $(x_1, \dots, x_n)$ to $(x_1, \dots, \widehat{x_i}, \dots, x_n)$.

Connections

The upper coconnections are the maps

$$\gamma^i_+ : \square^n \to \square^{n-1}, \quad i \in \{1, \ldots, n-1\},$$

taking $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_i x_{i+1}, \ldots, x_n)$.

The lower coconnections are the maps

$$\gamma^i_- : \square^n \to \square^{n-1}, \quad i \in \{1, \ldots, n-1\},$$

taking $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_i + x_{i+1} - x_i x_{i+1}, \ldots, x_n)$.

The reversals are the maps

$$\rho^i : \square^n \to \square^n, \quad i \in \{1, \ldots, n\},$$

taking $(x_1, \ldots, x_n)$ to $(x_1, \ldots, 1 - x_i, \ldots, x_n)$.

Our formula for coconnections is different from the more popular one that uses the function $\min(x, y) : \square^2 \to \square^1$ instead of $xy : \square^2 \to \square^1$. Both choices already occur in Boardman and Vogt: the one we use has the advantage that it is polynomial, so interacts better with de Rham theory.

The theory of cubical sets has serious difficulties when connections are omitted: for example, the fundamental theorem of Moore that simplicial groups are fibrant is not true in the cubical setting in the absence of upper (or lower) connections.

We work with upper connections because xy is easier to write than $x + y - xy$.

Let $\Omega(\square^n)$ be the differential graded commutative algebra of polynomial coefficient differential forms on the cube $\square^n$:

$$\Omega(\square^n) = \mathbb{F}[x_1, \dots, x_n, dx_1, \dots, dx_n].$$

We have $\Omega(\square^n) \cong \Omega(\square^1)^{\otimes n}$. This algebra is contractible.

As n varies, these algebras assemble to form a cubical differential graded commutative algebra $\Omega(\square^\bullet)$. Embedded in this cubical cochain complex is the simplicial cubical complex of elementary forms: these are the differential forms

$$C(\square^n) \cong C(\square^1)^{\otimes n}.$$

We recall that $C(\square^1)$ is spanned by $\{1, x, dx\}$.

The operator $p_1^{\otimes n}$ projects from $\Omega(\square^n)$ to $C(\square^n)$.

De Rham's theorem for cubes

The de Rham theorem is much simpler for cubical sets than for simplicial sets. Consider the homotopy $s_n^{\square} : \Omega^i(\square^n) \to \Omega^{i-1}(\square^n)$ given by the formula

$$s_n^{\square} = \sum_{i=1}^{n} p_1^{\otimes i-1} \otimes s_1 \otimes \mathrm{Id}_1^{\otimes n-i-1} .$$

It is an easy exercise to check that

$$ds_n^{\square} + s_n^{\square} d = 1 - p_1^{\otimes n}.$$

I believe the following result is new: I may be mistaken, and would welcome any pointers to the literature.

Lemma

The sequence of homotopies $s_\bullet^{\square}$ is compatible with the cubical operations: it is an endomorphism of the cubical cochain complex $\Omega(\square^\bullet)$.

If $X_\bullet$ is a cubical set, the coend

$$\Omega(X) = \int^n \Omega(\square^n) \otimes X_n$$

calculates the cohomology (with coefficients in $\mathbb{F}$) of $X_\bullet$. The homotopy s descends to a homotopy $s^\square : \Omega^i(X) \to \Omega^{i-1}(X)$, with $ds^\square + s^\square d = \mathrm{Id} - p^\square$, where $p^\square$ is a projection to the complex of cubical cochains

$$C(X) = \int^n C(\square^n) \otimes X_n.$$

The homotopy $s_\bullet^\square$ is compatible with reversal, since s_1 is, but not with exchanges.

If L is an L_∞algebra, we define cubical analogues of the Maurer–Cartan simplicial set $\mathsf{MC}_\bullet(L)$, and of $\gamma_\bullet(L)$:

$$\mathsf{MC}^{\square}_\bullet(L) = \mathsf{MC}(\Omega(\square^\bullet) \otimes L)$$

and

$$\gamma^{\square}_\bullet(L) = \mathsf{MC}(\Omega(\square^\bullet) \otimes L, s^{\square}_\bullet).$$

Exact analogues of all of the results in the simplicial case hold in the cubical case. I have little intuition for cubical Kan complexes (but see Bandiera's talk at this conference), and would rather transfer this construction back into the simplicial world. To do this, I will apply cubical unstraightening.

Cubical unstraightening

The bridge between the cubical and simplicial worlds is provided by the cosimplicial cubical set $Q^\bullet$. A glimpse of $Q^\bullet$ appears in the work of Boardman and Vogt, and its triangulation appears in Lurie's work on unstraightening of ∞-presheaves on ∞-categories, or the Grothendieck construction. (Boardman and Vogt work with topological categories, instead of simplicial categories, and study what is now called the unstraightening functor.)

The cosimplicial cubical set $Q^\bullet$ itself was considered by Kapulkin, Lindsey and Wong. The cubical set Q^n is obtained from $\square^n$ by an equivalence relation. In terms of the geometric realization of $[0,1]^n$, this equivalence relation is generated by the equivalences

$$(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \sim (0, \dots, 0, x_{i+1}, \dots, x_n),$$

where $1 < i \leqslant n$.

In other words, the lower face $\mathbf{p}_0^i \square^n$ is collapsed to a face of codimension i in $\square^n$.

The easiest way to think about Q^n is as the quotient of $\square^n$ defined by the following map between the geometric realizations of $\square^n$ to Δ^n:

$$(x_1, \dots, x_n) \mapsto (x_1 \dots x_n, (1 - x_1)x_2 \dots x_n, \dots, (1 - x_{n-1})x_n, 1 - x_n).$$

The simplest way to remember this is by the formulas

$$t_0 + \cdots + t_{i-1} = x_i \dots x_n.$$

For example, when $n = 2$,

$$t_0 = x_1 x_2, \qquad t_0 + t_1 = x_2, \qquad t_0 + t_1 + t_2 = 1.$$

The cosimplicial structure of $Q^\bullet$ is now clear. The cofaces $\mathbf{p}^i : Q^{n-1} \to Q^n$ and codegeneracies $\sigma^i : Q^n \to Q^{n-1}$ are induced by the maps between cubes

$$\mathbf{p}^i = \begin{cases} \mathbf{p}_0^1, & i = 0, \\ \mathbf{p}_1^i, & 1 \leqslant i \leqslant n, \end{cases} \qquad \sigma^i = \begin{cases} \sigma^1, & i = 0, \\ \gamma_+^i, & 1 \leqslant i \leqslant n. \end{cases}$$

We may now convert the Kan cubical set $\gamma_\bullet^\square(L)$ into a simplicial set: the n-simplices are the maps from Q^n to $\gamma_\bullet^\square(L)$.

There is a more direct way to construct the resulting simplicial set, using the simplicial differential graded algebra $\Omega(Q^n)$ of polynomial coefficient differential forms on the cubical set Q^n. These are just the differential forms on the cube $\square^n$ whose restriction to each lower face $(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$ is the pullback of a differential form on the face $(0, \dots, 0, x_{i+1}, \dots, x_n)$ by the projection between these faces.

This is clearly a differential graded subalgebra of $\Omega(\square^n)$, it is contractible, and is preserved by the homotopy operator $s_n^\square$.

The cubical unstraightening of $\gamma_\bullet^\square(L)$ is the Kan complex

$$\gamma_\bullet^Q = \mathrm{MC}\Big(\Omega(Q^\bullet) \otimes L, s_\bullet^\square\Big).$$

☡ $\Omega(Q^n)$ is not a free differential graded commutative algebra, unlike $\Omega(\Delta^n)$ and $\Omega(\square^n)$.

Lemma

For $n \geqslant 2$, $\Omega(Q^n)$ is not a finitely generated module over $\Omega^0(Q^n)$.

Our goal is to prove the following theorem.

Theorem

There is a natural isomorphism of simplicial sets $\gamma_\bullet(L) \cong \gamma^Q_\bullet(L)$.

The isomorphism $f : \mathsf{MC}(L, h) \to \mathsf{MC}(M)$ permits us to identify $\gamma_\bullet(L)$ with $\mathsf{MC}(C_\bullet \otimes L)$, where $C_\bullet \otimes L$ is the simplicial L_∞-algebra obtained by applying homological perturbation theory to the simplicial contraction of Dupont from $\Omega_\bullet \otimes L$ to $C_\bullet \otimes L$.

In this way, we obtain a retraction of the simplicial set $\mathsf{MC}_\bullet(L)$ to $\gamma_\bullet(L)$. Since this simplicial map has the homotopy equivalence $\gamma_\bullet(L) \hookrightarrow \mathsf{MC}_\bullet(L)$ as a section, it is itself a homotopy equivalence.

Theorem

There is a natural isomorphism of simplicial sets $\gamma_\bullet(L) \cong \gamma^Q_\bullet(L)$.

Consider the natural transformation

$$\gamma_\bullet(L) = \mathrm{MC}(\Omega(\Delta^\bullet) \otimes L, s_\bullet) \hookrightarrow \mathrm{MC}_\bullet(L) = \mathrm{MC}(\Omega(\Delta^\bullet) \otimes L) \longrightarrow \mathrm{MC}^Q_\bullet(L) = \mathrm{MC}(\Omega(Q^\bullet)) \longrightarrow \mathrm{MC}(C(Q^\bullet) \otimes L) \cong \gamma^Q_\bullet(L).$$

The last of these arrows is the retraction coming from homological perturbation theory.

By induction on the length of the filtration of L, we are reduced to the case where L is abelian, i.e. a cochain complex. It suffices to observe that in this case, the morphism of complexes

$$C(\Delta^\bullet) \otimes L \to C(Q^\bullet) \otimes L$$

is an isomorphism.

Categories with weak equivalences

A category with weak equivalences is a category $\mathcal{C}$ together with a subcategory $\mathcal{W} \subset \mathcal{C}$ of weak equivalences, containing all isomorphisms, such that if gf and hg are weak equivalences in the diagram

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z,$$

then f, g and h are weak equivalences as well.

This axiom is called the 2-out-of-6 condition. If $\mathcal{C}$ is the category of sets and $\mathcal{W}$ is the subcategory of isomorphisms, it amounts to the statement that a morphism that is both a split monomorphism and a split epimorphism is an isomorphism.

Categories of fibrant objects

A category of fibrant objects $\mathcal{C}$ is a small category with weak equivalences $\mathcal{W}$ together with a subcategory $\mathcal{F} \subset \mathcal{C}$ of fibrations, satisfying the following axioms. Here, we refer to morphisms in $\mathcal{W} \cap \mathcal{F}$ as trivial fibrations.

- There exists a terminal object e in $\mathcal{C}$, and every morphism $f : X \to e$ with target e is a fibration.
- Pullbacks of fibrations are fibrations.
- Pullbacks of trivial fibrations are trivial fibrations.
- Every morphism $f : X \to Y$ has a factorization

$$\begin{array}{ccccc} & & P & & \\ & \overset{r}{\nearrow} & & \overset{q}{\searrow} & \\ X & & \xrightarrow[f]{\hspace{3em}} & & Y \end{array}$$

  where r is a weak equivalence and q is a fibration.

Lemma (Brown's Lemma)

The trivial fibrations of a category of fibrant objects determine the weak equivalences: a morphism is a weak equivalence if and only if it is the composition of a section of a trivial fibration with a trivial fibration.

Small categories of fibrant objects are the objects of a bicategory, whose morphisms are the exact functors, and whose 2-morphisms are the natural transformations of exact functors.

Definition

A functor $F : \mathcal{C} \to \mathcal{D}$ of categories of fibrant object is exact if it takes fibrations to fibrations, trivial fibrations to trivial fibrations, terminal objects to terminal objects, and pullbacks of fibrations to pullbacks.

By Brown's lemma, exact functors preserve weak equivalences.

The approximation property for exact functors

An exact functor $F : \mathcal{C} \to \mathcal{D}$ is a homotopy equivalence if it satisfies Cisinski's generalization of the Waldhausen approximation property:

- F reflects weak equivalences: if $f : X \to Y$ is a morphism of $\mathcal{C}$ such that
$$F(f) : F(X) \to F(Y)$$
is a weak equivalence, then f is a weak equivalence;
- if $f : X \to F(A)$ is a morphism of $\mathcal{D}$, there is a morphism $g : B \to A$ of $\mathcal{C}$ and weak equivalences $p : \tilde{X} \to X$ and $q : \tilde{X} \to F(B)$ of $\mathcal{D}$, such that the following diagram commutes:
$$\begin{array}{ccc} \tilde{X} & \xrightarrow{q} & F(B) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle F(g)} \\ X & \xrightarrow[f]{} & F(A) \end{array}$$

Waldhausen's approximation property is the special case where $p : \tilde{X} \to X$ is the identity.

The homotopy category $\mathrm{Ho}(\mathcal{C})$ of a category $\mathcal{C}$ with weak equivalences $\mathcal{W}$ is the category obtained by inverting the weak equivalences. (If $\mathcal{C}$ is a category of fibrant objects, it suffices by Brown's lemma to invert the trivial fibrations.)

Theorem (Cisinski)

An exact functor $F : \mathcal{C} \to \mathcal{D}$ is a homotopy equivalence if and only if the induced functor $\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ on homotopy categories is an equivalence of categories.

Categories of fibrant objects occur throughout geometry. For example, the category of Lie groupoids is a category of fibrant objects, whose trivial fibrations are the Morita equivalences.

Kan fibrations and k-groupoids

The category of Kan complexes is the basic example of a category of fibrant objects. We now recall its definition.

Let $\partial_i\Delta^n$, $0 \leqslant i \leqslant n$, be the ith face of the n-simplex, and let

$$\Lambda^n_i = \bigcup_{j\neq i} \partial_j\Delta^n \subset \Delta^n$$

be the horn: this is the union of all faces containing the ith vertex.

A Kan fibration $f : X \to Y$ of simplicial sets is a morphism such that for all $n > 0$ and $0 \leqslant i \leqslant n$, the function

$$X_n = \mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_i, X) \times_{\mathrm{Hom}(\Lambda^n_i, Y)} \mathrm{Hom}(\Delta^n, Y)$$

is surjective. The morphism f is k-truncated (in the sense of Duskin) if this function is a bijection for $n > k$. A simplicial set X is a Kan complex (or ∞-groupoid) if the morphism $X \to \Delta^0$ to the terminal object is a Kan fibration, and a k-groupoid if the morphism $X \to \Delta^0$ is k-truncated.

Let

$$\partial\Delta^n = \bigcup_{j=0}^{n} \partial_j\Delta^n \subset \Delta^n$$

be the boundary of the n-simplex. An acyclic fibration $f : X \to Y$ of simplicial sets is a morphism such that for all $n \geqslant 0$, the function

$$X_n = \mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\partial\Delta^n, X) \times_{\mathrm{Hom}(\partial\Delta^n, Y)} \mathrm{Hom}(\Delta^n, Y)$$

is surjective.

Theorem

Let $0 \leqslant k \leqslant \infty$. The category of k-groupoids is a category of fibrant objects. The fibrations are the Kan fibrations, and the trivial fibrations are the acyclic fibrations.

Some variants of this theorem:

- reduced k-groupoids (simplicial sets with a single vertex), that is, k-groups;
- the essentially small category of k-groupoids of cardinality $< \aleph$.

Let $0 \leqslant k \leqslant \infty$. An L_∞-algebra L is k-truncated if $L^i = 0$, $i < -k$. We say that a morphism $\mathbf{f} : L \to M$ of L_∞-algebras is a fibration if f_1 is surjective, and a trivial fibration if f_1 is a surjective quasi-isomorphism and $\pi_0(f) : \pi_0(L) \to \pi_0(M)$ is an isomorphism.

Theorem

The category of k-truncated L_∞-algebras is a category of fibrant objects.

Likewise, we say that a strict morphism $f : L \to M$ of L_∞-algebras is a fibration if it is surjective, and a trivial fibration if it is a surjective quasi-isomorphism and $\pi_0(f) : \pi_0(L) \to \pi_0(M)$ is an isomorphism.

Theorem

The category of strict morphisms between k-truncated L_∞-algebras is a category of fibrant objects. The inclusion of the category of strict morphisms of L_∞-algebras into the category of L_∞-algebras is an exact functor, and a homotopy equivalence.

The heart of the proof of the last statement is verification of the second condition of the approximation property: this is the familiar fact that a

Theorem

The nerve $\mathrm{MC}_\bullet$ *is an exact functor from the category of* k*-truncated* L_∞*-algebras to the category of* k*-groupoids.*

Theorem

The functor $\gamma_\bullet$ *is an exact functor from the category of strict morphisms of* k*-truncated* L_∞*-algebras to the category of* k*-groupoids.*

The proofs that the functors $\mathrm{MC}_\bullet(L)$ and $\gamma_\bullet(L)$ take fibrations to Kan fibrations is by induction in the length of the filtration of L: this is the main place in the theory where (pro-)nilpotence is essential.

Deligne 2-groupoid

If $\mathbf{g}$ is a differential graded Lie algebra concentrated in degrees $[-1, \infty)$ corresponding to an L_∞-algebra L concentrated in degrees $[-2, \infty)$, the Kan complex $\gamma_\bullet(L)$ is the (Duskin) nerve of a strict 2-groupoid.

On the other hand, Deligne associated an explicit strict 2-groupoid to $\mathbf{g}$, whose objects x are the Maurer–Cartan elements, whose morphisms are pairs (x, y), where $y \in \mathbf{g}^0$, and whose 2-morphisms are triples (x, y, z), where $z \in \mathbf{g}^{-1}$. This construction appears in reference [2].

For both of these 2-groupoids, π_0 is the set of gauge equivalence classes of Maurer–Cartan elements (the coarse moduli space of the 2-stack, in geometric language).

At a Maurer–Cartan element x, π_1 is the isotropy group of x: the quotient of the group of gauge transformations preserving x by equivalence under the action of 2-morphisms.

π_2 is the space of Casimir elements: the center of the Lie bracket $\{a, b\}_x = [[x, a], b]$ on $\mathbf{g}^{-1}$.

Thus, one could show that they are isomorphic by constructing a map between them. This is surprisingly difficult: for one construction, see reference [4].

In his talk at this workshop, Bandiera shows that the cubical set $\gamma^c(L)$ is the nerve of the Deligne 2-groupoid. Combined with our main theorem, this establishes that the Deligne 2-groupoid is isomorphic to $\gamma_\bullet(L)$.

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