Rational homotopy of operads. Models of mapping spaces and applications

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Introduction

- Motivations: Applications of operads to the study of embedding spaces $\mathtt{Emb}(M, N)$ & Grothendieck–Teichmüller groups
- Fundamental case: Take $M = \mathbb{R}^m$, $N = \mathbb{R}^n$, $i_m : \mathbb{R}^m \hookrightarrow \mathbb{R}^n$ the standard embedding $i_m : (x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$, and:

$$\mathtt{Emb}_c(\mathbb{R}^m, \mathbb{R}^n) := \{f : \mathbb{R}^m \hookrightarrow \mathbb{R}^n \,|\, \exists K \text{ compact with } f\,|_{\mathbb{R}^m \setminus K} = i_m\},$$
$$\mathtt{Imm}_c(\mathbb{R}^m, \mathbb{R}^n) := \{f : \mathbb{R}^m \looparrowright \mathbb{R}^n \,|\, \exists K \text{ compact with } f\,|_{\mathbb{R}^m \setminus K} = i_m\},$$
$$\overline{\mathtt{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) := \mathrm{hofib}(\mathtt{Emb}_c(\mathbb{R}^m, \mathbb{R}^n) \to \mathtt{Imm}_c(\mathbb{R}^m, \mathbb{R}^n)).$$

- Theorem (recollections): We have homotopy equivalences

$$\overline{\mathtt{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) \xrightarrow[(1)]{\sim} \mathtt{Map}^h_{\mathsf{D}_m\,\mathcal{B}i\mathcal{M}od}(\mathsf{D}_m, \mathsf{D}_n)$$

$$\overline{\mathtt{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) \xrightarrow[(2)]{\sim} \Omega^{m+1}\,\mathtt{Map}^h_{\mathcal{T}op\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n)$$

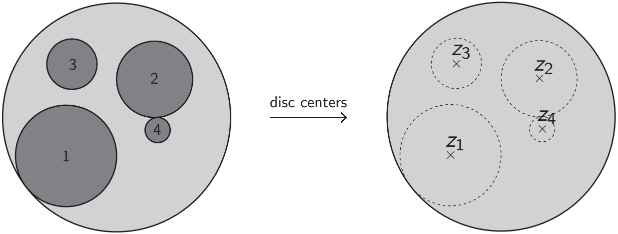
$$\mathtt{Map}^h_{\mathsf{D}_m\,\mathcal{B}i\mathcal{M}od}(\mathsf{D}_m, \mathsf{D}_n) \xrightarrow[(3)]{\sim} \Omega^{m+1}\,\mathtt{Map}^h_{\mathcal{T}op\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n)$$

as soon as $n - m \geq 3$, where D_m is the operad of little m-discs.

  - (1) was obtained by Sinha (for $m = 1$) and by Arone-Turchin (for all $m \geq 1$).
  - (2) was obtained by Boavida-Weiss (for all $m \geq 1$).
  - (3) was obtained by Dwyer-Hess (for $m = 1$) and by Ducoulombier-Turchin (for all $m \geq 1$).

Recollections on the operads of little discs

- The little n-discs spaces $\mathrm{D}_n(r)$ consist of collections of r little n-discs with disjoint interiors inside a fixed unit n-disc $\mathbb{D}^n$ (see Figure).



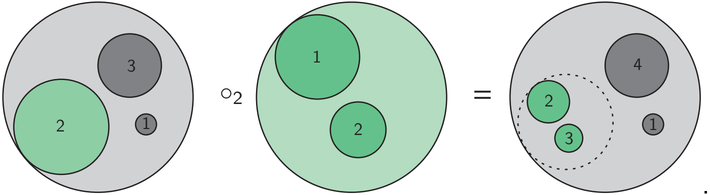
$\in \mathrm{D}_2(4)$ $\in \mathrm{F}(\mathring{\mathbb{D}}^2, 4)$

- The configuration spaces $\mathrm{F}(\mathring{\mathbb{D}}^n, r)$ consist of collections of r distinct points in the open disc $\mathring{\mathbb{D}}^n$ (see Figure).
- There is an obvious homotopy equivalence $\mathrm{D}_n(r) \xrightarrow{\sim} \mathrm{F}(\mathring{\mathbb{D}}^n, r)$.

- The symmetric group Σ_r acts on $\mathsf{D}_n(r)$ by permutation of the little disc indices (and on the configuration space similarly).
- The little n-discs spaces (unlike the configuration spaces) inherit operadic composition operations

$$\circ_i : \mathsf{D}_n(k) \times \mathsf{D}_n(l) \to \mathsf{D}_n(k+l-1)$$

given by the following substitution process



- *The little n-discs operad D_n is the object defined by the collection of spaces $\mathsf{D}_n(r)$ together with these structure operations.*

- Theorem (F. Cohen): For $n \geq 2$, we have an identity:

$$\mathtt{H}_*(\mathsf{D}_n) = \mathsf{Pois}_n,$$

where Pois_n is the operad of n-Poisson algebras, with:

$$x_1 x_2 = [\mathrm{pt}] \in \mathtt{H}_0(\mathsf{D}_n(2)), \quad [x_1, x_2] = [\mathbb{S}^{n-1}] \in \mathtt{H}_{n-1}(\mathsf{D}_n(2)),$$

so that:

$$\begin{aligned}
\mathtt{H}_*(\mathsf{D}_n(2)) &= \mathbb{Q}\, x_1 x_2 \oplus \mathbb{Q}[x_1, x_2], \\
\mathtt{H}_*(\mathsf{D}_n(3)) &= \mathbb{Q}\, x_1 x_2 x_3 \\
&\quad \oplus \mathbb{Q}[x_1, x_2] x_3 \oplus \mathbb{Q}[x_1, x_3] x_2 \oplus \mathbb{Q}\, x_1 [x_2, x_3] \\
&\quad \oplus \mathbb{Q}[[x_1, x_2], x_3] \oplus \mathbb{Q}[[x_1, x_3], x_2], \\
\mathtt{H}_*(\mathsf{D}_n(4)) &= \dots.
\end{aligned}$$

- Theorem (V. Arnold, F. Cohen): For $n \geq 2$, we have an identity:

$$\mathrm{H}^*(\mathsf{D}_n(r)) = \mathrm{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r)) = \frac{\bigwedge(\omega_{ij}, 1 \leq i \neq j \leq r)}{(\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij})}$$

where $\omega_{ij} = \pi_{ij}^*(\omega_{\mathbb{S}^{n-1}})$, and cooperad structure operations

$$\circ_i^* : \mathrm{H}^*(\mathsf{D}_n)(\underline{r}) \to \mathrm{H}^*(\mathsf{D}_n)(\underline{r} / \underline{l}) \otimes \mathrm{H}^*(\mathsf{D}_n)(\underline{l})$$

such that

$$\circ_i^*(\omega_{ab}) = \begin{cases} \omega_{ab} \otimes 1 & \text{for } a, b \notin \underline{l}, \\ \omega_{ai} \otimes 1 & \text{for } a \notin \underline{l},\ b \in \underline{l}, \\ \omega_{ib} \otimes 1 & \text{for } a \in \underline{l},\ b \notin \underline{l}, \\ 1 \otimes \omega_{ab} & \text{for } a, b \in \underline{l}, \end{cases}$$

for $\underline{r} = \{1, \ldots, k + l - 1\}$, $\underline{l} = \{i, \ldots, i + l - 1\} \subset \underline{r}$, and using $\underline{r} / \underline{l} \simeq \{1, \ldots, i, \ldots, k\}$ and $\underline{l} \simeq \{1, \ldots, l\}$.

Main objectives and results

- General goal: Give a combinatorial (graph complex) description of $\mathtt{Map}^h_{\mathcal{T}op\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n^{\mathbb{Q}})$, and of $\mathtt{Aut}^h_{\mathcal{T}op\mathcal{O}p}(\mathsf{D}_n^{\mathbb{Q}})$, where $\mathsf{D}_n^{\mathbb{Q}}$ is a rationalization of D_n.
- Remark: we have

$$\mathtt{Map}^h_{\mathcal{T}op\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n^{\mathbb{Q}}) \sim \mathtt{Map}^h_{\mathcal{T}op\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n)^{\mathbb{Q}}$$

as soon as $n - m \geq 3$. The obtained description accordingly gives results on the rational homotopy of the space of embeddings.

- Theorem A (BF-Turchin-Willwacher): For $n \geq m \geq 2$, we have:

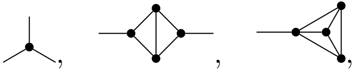
$$\mathtt{Map}^h_{\mathcal{T}op\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n^{\mathbb{Q}}) \sim \mathtt{MC}_\bullet(\mathsf{HGC}_{mn}),$$

  where HGC_{mn} is the hairy graph complex. This relation extends to the case $n > m = 1$ with HGC_{1n} equipped with the Shoikhet L_∞-structure.
- Corollary:
  - For any $n \geq m \geq 2$ (or $n > m = 1$), we have the identity:

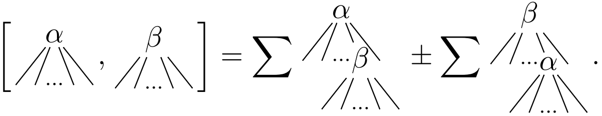
$$\pi_*(\mathtt{Map}^h_{\mathcal{T}op\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n^{\mathbb{Q}}), \omega) = H_{*-1}(\mathsf{HGC}^\omega_{mn}),$$

    for any $\omega \in \mathtt{MC}_0(\mathsf{HGC}_{mn})$, where HGC^ω_{mn} is the complex HGC_{mn} equipped with the twisted differential
    $\delta_\omega = \delta + [\omega, -] + (\text{extra terms in the } L_\infty\text{-case})$.

- The hairy graph complex: The complex HGC_{mn} is a complete Lie dg-algebra which consists of connected graphs with internal vertices $\bullet$, internal edges, and external legs (the hairs), such as:



  The (homological) degree of a hairy graph is determined by counting $\deg(\bullet) := -n$ for each vertex, $\deg(\bullet\!-\!\bullet) := n-1$ for each internal edge, $\deg(\bullet\!-) = n-1-m$ for each hair, and by adding a global degree shift by m.
- The differential is defined by the blow-up of internal vertices.
- The Lie bracket is given by:



- Theorem B (BF-Turchin-Willwacher, BF-Willwacher): For $n \geq 2$, we have a weak homotopy equivalence of simplicial monoids

$$\mathtt{Aut}^h_{\mathcal{T}op\mathcal{O}p}(\mathsf{D}_n^{\mathbb{Q}}) \sim \mathbb{Q}^\times \ltimes \mathtt{Z}_\bullet(\mathsf{GC}_n^2),$$

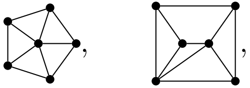
where GC_n^2 denotes the Kontsevich graph complex (with bivalent vertices allowed), and $\mathtt{Z}_\bullet(\mathsf{GC}_n^2) := \mathtt{Z}^0(\mathsf{GC}_n^2 \hat{\otimes} \Omega^*(\Delta^\bullet))$ is equipped with a monoid structure deduced from the BCH formula.

- Remark: In the case $n = 2$, we have $\mathtt{H}_*(\mathsf{GC}_2^2) = \mathbb{Q}[1] \oplus \mathfrak{grt}_1$ (Willwacher), where $\mathfrak{grt}_1$ is the graded Grothendieck–Teichmüller Lie algebra, and this result reflects the relation:

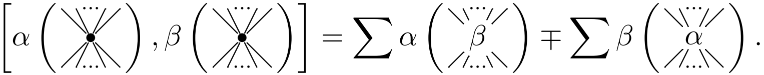
$$\mathtt{Aut}^h_{\mathcal{T}op\mathcal{O}p}(\mathsf{D}_2^{\mathbb{Q}}) \sim \mathtt{GT}(\mathbb{Q}) \ltimes \mathtt{SO}(2)^{\mathbb{Q}},$$

where $\mathtt{GT}(\mathbb{Q})$ is the Grothendieck–Teichmüller group (BF, with profinite generalizations by Horel and by Boavida-Horel-Robertson).

- The Kontsevich graph complex: The complex GC_n^2 consists of connected graphs of the form



and where we take the grading such that $\deg(\bullet) = -n$, $\deg(\bullet\!-\!\bullet) = n - 1$.
- The differential is defined by the blow-up of internal vertices again.
- The Lie bracket is given by:



Plan

§1. Introduction to the rational homotopy of operads

Quick recollections on Sullivan's models:

- The model is given by Sullivan's functor of PL differential forms $\Omega^* : s\mathcal{S}et^{op} \to dg\,\mathcal{C}om$. For a simplex $\Delta^n = \{0 \leq x_1 \leq \cdots \leq x_n \leq 1\}$, we have:

$$\Omega^*(\Delta^n) = \mathbb{Q}[x_1, \ldots, x_n, dx_1, \ldots, dx_n].$$

- This functor has a left adjoint $\mathtt{G}_\bullet : dg\,\mathcal{C}om \to s\mathcal{S}et^{op}$ such that $\mathtt{G}_n(A) = \mathtt{Mor}_{dg\,\mathcal{C}om}(A, \Omega^*(\Delta^n))$, for each $n \in \mathbb{N}$. Let:

$$\langle A \rangle := \text{derived functor of } \mathtt{G}(A) = \mathtt{Mor}_{dg\,\mathcal{C}om}(R_A, \Omega^*(\Delta^\bullet)),$$

where $R_A \xrightarrow{\sim} A$ is any cofibrant resolution of A in $dg\,\mathcal{C}om$.

- If X satisfies reasonable finiteness and nilpotence assumptions, then

$$X^{\mathbb{Q}} := \langle \Omega^*(X) \rangle$$

defines a rationalization of the space X.

- Idea: Take the category of cooperads in commutative dg-algebras (the category of Hopf dg-cooperads) as a model for the category of operads in simplicial sets (and in topological spaces).
- Theorem (BF):
  - We have a Quillen pair $\mathtt{G}_\bullet : dg\,\mathcal{H}opf\,\mathcal{O}p^c \rightleftarrows s\mathcal{S}et\mathcal{O}p^{op} : \Omega^*_\sharp$ where
    $$\mathtt{G}_\bullet(\mathsf{A})(r) = \mathtt{G}_\bullet(\mathsf{A}(r))$$
    and $\Omega^*_\sharp : \mathsf{P} \mapsto \Omega^*_\sharp(\mathsf{P})$ is an operadic upgrading of the Sullivan functor of PL forms.
  - Let R be a cofibrant operad such that $\dim \mathtt{H}^*(\mathsf{R}(r)) < \infty$ for each r. Then we have a quasi-isomorphism of dg-algebras
    $$\Omega^*_\sharp(\mathsf{R})(r) \xrightarrow{\sim} \Omega^*(\mathsf{R}(r))$$
    in each arity r.
  - If we set $\mathsf{R}^{\mathbb{Q}} := \langle \Omega^*_\sharp(\mathsf{R}) \rangle$, then we have $\mathsf{R}^{\mathbb{Q}}(r) \sim \mathsf{R}(r)^{\mathbb{Q}}$ for each arity r.

- Remark: The adjunction relations imply that giving a morphism of Hopf dg-cooperads

$$\phi_\sharp : \mathsf{A} \to \Omega^*_\sharp(\mathsf{P}),$$

for P an operad in simplicial sets, is equivalent to giving a collection of morphisms of commutative dg-algebras

$$\phi : \mathsf{A}(r) \to \Omega^*(\mathsf{P}(r))$$

that:

  - preserve the action of the symmetric group
  - and make commute the diagrams

$$\begin{array}{ccc}
\mathsf{A}(k+l-1) & \xrightarrow{\phi} & \Omega^*(\mathsf{P}(k+l-1)) \\
 & & \big\downarrow{\scriptstyle \circ_i^*} \\
{\scriptstyle \circ_i^*}\big\downarrow & & \Omega^*(\mathsf{P}(k)\times\mathsf{P}(l)) \\
 & & \big\uparrow{\scriptstyle \sim} \\
\mathsf{A}(k)\otimes\mathsf{A}(l) & \xrightarrow{\phi\otimes\phi} & \Omega^*(\mathsf{P}(k))\otimes\Omega^*(\mathsf{P}(l))
\end{array}$$

- Theorem (BF-Willwacher): The model category of Hopf dg-cooperads is equipped with a simplicial enrichment such that:

$$\mathtt{Map}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{A},\mathsf{B}) = \mathtt{Mor}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c\,/\,\Omega^*(\Delta^\bullet)}(\mathsf{A}\otimes\Omega^*(\Delta^\bullet),\mathsf{B}\otimes\Omega^*(\Delta^\bullet)).$$

- Corollary: For R a cofibrant operad in $\mathcal{T}op$ (satisfying $\dim\mathtt{H}^*(\mathsf{R}(r))<\infty$), we have:

$$\mathtt{Aut}^h_{\mathcal{T}op\mathcal{O}p}(\mathsf{R}^{\mathbb{Q}}) \sim \mathtt{Aut}^h_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\Omega^*_\sharp(\mathsf{R}))$$

with $\mathtt{Aut}^h_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(-)$ deduced from this simplicial enrichment.

§2. Formality and graph complex models of the little discs operads

- Theorem (BF-Willwacher, Kontsevich): We have a zigzag of quasi-isomorphisms of Hopf dg-cooperads

$$\mathsf{Pois}_n^c \xleftarrow{\sim} \cdot \xrightarrow{\sim} \mathtt{R}\,\Omega_\sharp^*(\mathsf{D}_n),$$

  where:
  - $\mathsf{Pois}_n^c := \mathtt{Hom}(\mathsf{Pois}_n, \mathbb{Q}) = \mathtt{Hom}(\mathtt{H}_*(\mathsf{D}_n), \mathbb{Q}) = \mathtt{H}^*(\mathsf{D}_n)$,
  - $\mathtt{R}\,\Omega_\sharp^*(\mathsf{D}_n) :=$ derived functor of $\Omega_\sharp^*(\mathsf{D}_n)$.
- Corollary:

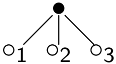
$$\begin{aligned}\mathtt{Map}^h_{\mathcal{T}op\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n^{\mathbb{Q}}) &\sim \mathtt{Map}^h_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathtt{R}\,\Omega_\sharp^*(\mathsf{D}_n), \mathtt{R}\,\Omega_\sharp^*(\mathsf{D}_m)) \\ &\sim \mathtt{Map}^h_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{Pois}_n^c, \mathsf{Pois}_m^c).\end{aligned}$$

- Remark: Kontsevich's construction involves a cooperad of graphs as middle term.

The graph cooperad (1)

- The components of the cooperad of graphs $\mathsf{Graphs}_n^c(r)$ are spanned by graphs with internal vertices $\bullet$ and external vertices $\circ_i$ indexed by $i = 1, \dots, r$, and which fulfill the following assumptions:
  1. loops $\circlearrowleft$ (edges with the same origin and endpoint) are not allowed,
  2. the internal vertices $\bullet$ are at least trivalent,
  3. each internal vertex $\bullet$ is connected to an external vertex $\circ_i$ by a path of edges in the graph.

  For instance, we have:

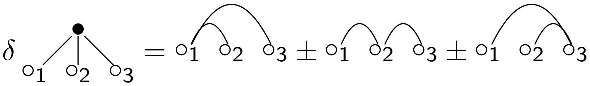
  

  $\in \mathsf{Graphs}_n^c(3)$.

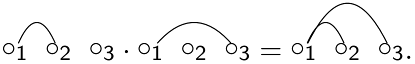
- The (cohomological) degree of a graph is defined by counting $\deg^*(\bullet) = -n$ for each internal vertex $\bullet$ and $\deg^*(-) = n - 1$ for each edge $-$.

The graph cooperad (2)

- The differential of graphs is defined by merging internal vertices together or by merging internal vertices and external vertices. For instance:



- The product of graphs is given by the union along external vertices. For instance:

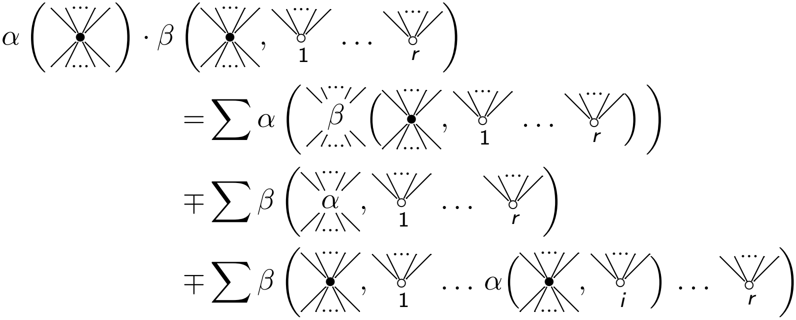


- The cooperad coproducts $\circ_i^* : \mathrm{Graphs}_n^c(k+l-1) \to \mathrm{Graphs}_n^c(k) \otimes \mathrm{Graphs}_n^c(l)$ are defined by collapsing subgraphs based at the external vertices indexed by $i, \dots, i+l-1$ onto an external vertex:

$$\circ_*(\gamma) = \sum_{\alpha \subset \gamma} \gamma/\alpha \otimes \alpha.$$

The graph cooperad (3)

- Variation: Let Graphs_n^{2c} be a variant of Graphs_n^c with bivalent internal vertices allowed. We have $\mathsf{Graphs}_n^c \xrightarrow{\sim} \mathsf{Graphs}_n^{c2}$.
- Observation: The Lie algebra GC_n^2 acts on Graphs_n^{2c} through morphisms of Hopf dg-cooperads. For the dual operad in dg-modules $\mathsf{Graphs}_n^2(r) = \mathtt{Hom}(\mathsf{Graphs}_n^{2c}(r), \mathbb{Q})$ this action reads:



- **Theorem (Kontsevich, BF-Willwacher):** We have a quasi-isomorphism of Hopf dg-cooperads

$$\mathsf{Graphs}_n^c \xrightarrow{\sim} \mathrm{R}\,\Omega_\sharp^*(\mathsf{D}_n)$$

  such that $\circ_1 \frown \circ_2 \mapsto \omega_{\mathbb{S}^{n-1}}$.
- **Proposition:** We have a quasi-isomorphism of Hopf dg-cooperads

$$\mathsf{Graphs}_n^c \xrightarrow{\sim} \mathrm{H}^*(\mathsf{D}_n) = \mathsf{Pois}_n^c$$

  such that $(1) \frown (2) \mapsto \omega_{12}$.
- **Observation:** The graph cooperad Graphs_n^c is cofibrant in $dg\,\mathcal{H}opf\,\mathcal{O}p^c$.

§3. Ideas in the proofs of Theorem A-B

(1) Applications of cofibrant and fibrant resolutions

- Proposition: We have

$$\mathtt{Map}^h_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{Graphs}^c_n, \mathsf{Pois}^c_m) = \mathtt{Map}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{Graphs}^c_n, \mathtt{W}^c(\mathsf{Pois}^c_m)),$$

  where $\mathtt{W}^c(-)$ is an analogue of the Boardman-Vogt W-construction for Hopf dg-cooperads.

- Proposition: We have

$$\mathtt{Map}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{Graphs}^c_n, \mathtt{W}^c(\mathsf{Pois}^c_m)) \sim \mathtt{MC}_\bullet(\mathtt{BiDer}(\mathsf{Graphs}^c_n, \mathtt{W}^c(\mathsf{Pois}^c_m))),$$

  for some natural L_∞-algebra structure on $\mathtt{BiDer}(\mathsf{Graphs}^c_n, \mathtt{W}^c(\mathsf{Pois}^c_m))$.

(2) Proof of Theorem A

- We have

$$\mathtt{BiDer}(\mathsf{Graphs}_n^c, \mathtt{W}^c(\mathsf{Pois}_m^c)) \sim \mathtt{Hom}(I\,\mathsf{Graphs}_n^c, \mathtt{B}^c(\mathsf{Pois}_m^c)),$$

  where $I\,\mathsf{Graphs}_n^c(r)$ is the complex of internally connected graphs inside $\mathsf{Graphs}_n^c(r)$ and $\mathtt{B}^c(-)$ is the operadic cobar construction.

- We moreover have $\mathtt{B}^c(\mathsf{Pois}_m^c) \sim \Lambda^m\,\mathsf{Pois}_m$ by the Koszul self duality of the operad Pois_m.
- We have a natural map $\mathsf{HGC}_{mn} \to \mathtt{Hom}(I\,\mathsf{Graphs}_n^c, \Lambda^m\,\mathsf{Pois}_m)$ which yields a quasi-isomorphism of L_∞-algebras

$$\mathtt{BiDer}(\mathsf{Graphs}_n^c, \mathtt{W}^c(\mathsf{Pois}_m^c)) \sim \mathsf{HGC}_{mn},$$

  and the conclusion of Theorem A follows. □

(3) Proof of Theorem B

- For a connected component $\mathtt{Map}(-,-)_\psi \subset \mathtt{Map}(-,-)$, we have :

$$\mathtt{Map}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{Graphs}_n^c, \mathtt{W}^c(\mathsf{Pois}_m^c))_\psi \sim \mathtt{MC}_\bullet(\mathtt{BiDer}(\mathsf{Graphs}_n^c, \mathtt{W}^c(\mathsf{Pois}_m^c))^\omega)_0 \sim \mathtt{MC}_\bullet(\mathsf{HGC}_{nm}^\omega)_0,$$

with $\omega \in \mathtt{MC}_0(\mathsf{HGC}_{nm})$ corresponding to $\psi : \mathsf{Graphs}_n^c \to \mathtt{W}^c(\mathsf{Pois}_m^c)$.

  - For $m = n$ and ψ corresponding to the identity map on D_n, we have a further reduction

$$\Sigma^{-1}(\mathbb{Q} \oplus \mathsf{GC}_n^2) \xrightarrow{\sim} \mathsf{HGC}_{nn}^{2\omega},$$

    with GC_n^2 regarded as a trivial Lie algebra, and HGC_{nn}^2 is the variant of HGC_{nn} where bivalent vertices are allowed.
  - The mapping $\mathtt{Z}_\bullet(\mathsf{GC}_n^2) \to \mathtt{Map}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{Graphs}_n^{2c}, \mathsf{Graphs}_n^{2c})_{\mathrm{id}}$ yielded by the action of the Lie algebra GC_n^2 on Graphs_n^{2c} can be prolonged to a weak-equivalence :

$$\mathtt{Z}_\bullet(\mathsf{GC}_n^2) \xrightarrow{\sim} \mathtt{Map}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{Graphs}_n^{2c}, \mathtt{W}^c(\mathsf{Pois}_n^{2c}))_\psi,$$

    from which the result of Theorem B follows. □

§4. Conclusion

- Definition of graph complexes in positive characteristic?
- Combinatorial (graph complex) description of $\mathtt{Map}^h_{\mathcal{T}op\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n^\wedge)$, and of $\mathtt{Aut}^h_{\mathcal{T}op\mathcal{O}p}(\mathsf{D}_n^\wedge)$, where $\mathsf{D}_n^\wedge$ is the p-completion of D_n?

§A. Labelled hairy graphs and the generalization of the model

- Context: We consider the case $\overline{\mathtt{Emb}}_c(M, \mathbb{R}^n)$ for $M \subset \mathbb{R}^m$ an open subset which contains a neighbourhood of ∞ in $\mathbb{R}^m$.
- Theorem (recollections): We have

$$\overline{\mathtt{Emb}}_c(M, \mathbb{R}^n) \sim \mathtt{Map}^h_{\mathsf{D}_m\,\mathcal{B}i\mathcal{M}od}(\mathsf{D}_M, \mathsf{D}_n),$$

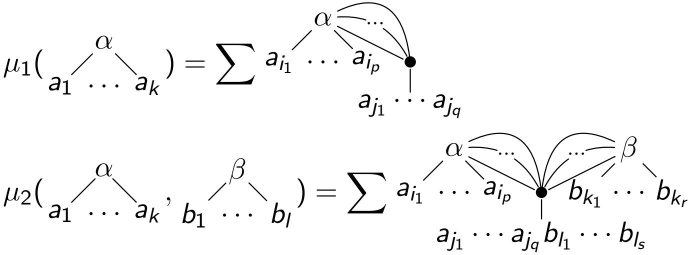
where $\mathsf{D}_M(r) = \mathtt{Emb}^{st}(\coprod^r \mathbb{D}^m, M)$. (Follows from the results of Arone-Turchin.)

- Objective: Give a combinatorial description of $\mathtt{Map}^h_{\mathsf{D}_m\,\mathcal{B}i\mathcal{M}od}(\mathsf{D}_M, \mathsf{D}_n^{\mathbb{Q}})$.
- Remark: We now have

$$\mathtt{Map}^h_{\mathsf{D}_m\,\mathcal{B}i\mathcal{M}od}(\mathsf{D}_M, \mathsf{D}_n^{\mathbb{Q}})_f \sim \mathtt{Map}^h_{\mathsf{D}_m\,\mathcal{B}i\mathcal{M}od}(\mathsf{D}_M, \mathsf{D}_n)_f^{\mathbb{Q}},$$

for any $f : \mathsf{D}_M \to \mathsf{D}_n$, where $\mathtt{Map}(-,-)_f$ denotes the connected component of the mapping spaces associated to such a map f, and the map $\pi_0 \mathtt{Map}^h_{\mathsf{D}_m\,\mathcal{B}i\mathcal{M}od}(\mathsf{D}_M, \mathsf{D}_n) \to \pi_0 \mathtt{Map}^h_{\mathsf{D}_m\,\mathcal{B}i\mathcal{M}od}(\mathsf{D}_M, \mathsf{D}_n^{\mathbb{Q}})$ is finite-to-one.

- The decorated hairy graph complex HGC_{An}, where A is an augmented commutative dg-algebra, consists of graphs with internal vertices $\bullet$, internal edges, and external legs (the hairs) decorated by elements of $\bar{A}$. The differential is defined by the blow-up of vertices again.
- The complex HGC_{An} is equipped with an L_∞-structure given by the operations $\mu_1, \mu_2, \mu_3, \dots$ such that:



…

- Theorem (BF-Turchin-Willwacher): Let A be a Sullivan model of $\hat{M} = M \cup \{\infty\}$. For $n \geq m \geq 2$, we have:

$$\mathtt{Map}^h_{\mathsf{D}_m\,\mathcal{B}i\mathcal{M}od}(\mathsf{D}_M, \mathsf{D}_n^{\mathbb{Q}}) \sim \mathtt{MC}_\bullet(\mathsf{HGC}_{An}),$$

  where HGC_{An} is the decorated hairy graph complex.
- Example of application: The set $\pi_0\overline{\mathtt{Emb}}(\mathbb{S}^3 \times \mathbb{S}^3, \mathbb{R}^{11})$ is finite.
  - Indeed, we have $\overline{\mathtt{Emb}}(\mathbb{S}^3 \times \mathbb{S}^3, \mathbb{R}^{11})^{\mathbb{Q}} \sim \mathtt{MC}_\bullet(\mathsf{HGC}_{H^*(\mathbb{S}^3)\times H^*(\mathbb{S}^3),11})$ and $\mathsf{HGC}_{H^*(\mathbb{S}^3)\times H^*(\mathbb{S}^3),11}$ is null in degree 1.)

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