Cluster algebras, Higgs categories and braid group actions

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Université Paris Cité

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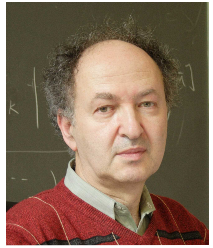
GAP XX
RIMS Kyoto
April 16-17, 2026

Plan

Plan



George Lusztig

Can. bases / Total pos.
George Lusztig
early 90s

Cluster algebras
Fomin-Zelevinsky
2001

S. Fomin

A. Zelevinsky, 1953-2013

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triang. 2-CY cat.
Buan-Marsh-Reineke-Reiten-Todorov '04

additive categorif.

Kang-Kashiwara-Kim-Oh
'14, '15

Qin '15

repres. of quantum affine algebras
Hernandez-Leclerc '09

monoidal categorif.

discrete dyn. syst.
Fomin-Zel. '01
K '10

Poisson geometry
Gekhtman-Shapiro-Vainshtein '02

Higher Teichm. th.
Fock-Goncharov '03

DT-theory
Konts.-Soib. '08
Nagao '10

mirror sym.
Gross-Hacking-Keel-Konts. '14
(Argüz-Bousseau '22)

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sc. amplitudes
Arkani-Hamed & Trnka '13

String theory
BPS state counts
Gaiotto-Moore-Neitzke '08

Past fruitful interaction

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Motivation for this minicourse

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$\boxed{\text{Cluster algebras Fomin-Zelevinsky 2001}}$

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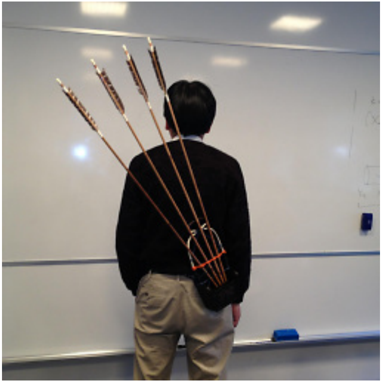
The basic combinatorial input

Cluster algebras (without coefficients)
are associated with
quivers.

A quiver is a container for arrows



A quiver is an oriented graph

Definition

A *quiver* Q is an oriented graph: it is given by

- a set Q_0 (the set of vertices)
- a set Q_1 (the set of arrows)
- two maps
  - $s : Q_1 \to Q_0$ (sends an arrow to its source)
  - $t : Q_1 \to Q_0$ (sends an arrow onto its target)

Remark

A quiver is a ‘category without composition’.

A quiver may have loops and cycles.

Example

$$
Q: \quad
\begin{tikzcd}
 & 3 \arrow[dl, "\lambda"'] & & 5 \arrow[loop left, "\alpha"] \arrow[rr, shift left=2] \arrow[rr] \arrow[rr, shift right=2] & & 6 \\
1 \arrow[rr, "\nu"'] & & 2 \arrow[ul, "\mu"'] \arrow[rr, shift left, "\beta"] & & 4 \arrow[ll, shift left, "\gamma"] & 7 \arrow[u]
\end{tikzcd}
$$

We have $Q_0 = \{1, 2, 3, 4, 5, 6, 7\}$, $Q_1 = \{\alpha, \beta, \ldots\}$.
α is a *loop*, (β, γ) is *2-cycle*, 7 is a *source* of Q, 6 is a sink of Q.

Quiver mutation

Let Q be a finite quiver without loops nor 2-cycles with $Q_0 = \{1, \ldots, n\}$ (always assumed from now on).

Definition (Fomin–Zelevinsky)

Let $j \in Q_0$. The *mutated quiver* $\mu_j(Q)$ is the quiver obtained from Q as follows

1) for each subquiver $i \xrightarrow{\beta} j \xrightarrow{\alpha} k$, add a new arrow
   $i \xrightarrow{[\alpha\beta]} k$;
2) reverse all the arrows with source or target j;
3) remove all 2-cycles (e.g. $\bullet \rightleftarrows\!\!\rightarrow \bullet$ yields $\bullet \longrightarrow \bullet$).

Simple examples of quiver mutation

A simple example:

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 2 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd}$$

1)

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 2 \arrow[ur] \arrow[rr, shift left] & & 3 \arrow[ll, shift left] \end{tikzcd}$$

2)

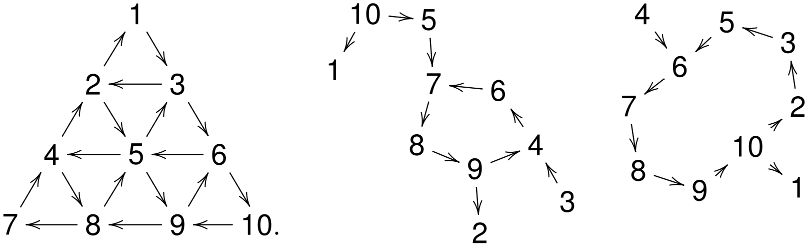
$$\begin{tikzcd} & 1 \arrow[dl] & \\ 2 \arrow[rr, shift left] & & 3 \arrow[ul] \arrow[ll, shift left] \end{tikzcd}$$

3)

$$\begin{tikzcd} & 1 \arrow[dl] & \\ 2 & & 3 \arrow[ul] \end{tikzcd}$$

More complicated examples: use a computer!

https://webusers.imj-prg.fr/~bernhard.keller/quivermutation



Recall: We wanted to define cluster algebras!

Mutation of seeds

Definition

A seed is a pair (R, u), where

a) R is a quiver with n vertices;

b) $u = \{u_1, \ldots, u_n\}$ is a a free gen. set for the field $\mathbb{Q}(x_1, \ldots, x_n)$.

Example: $(1 \to 2 \to 3, \{x_1, x_2, x_3\}) = (x_1 \to x_2 \to x_3)$.

Definition

For a vertex j of R, the mutated seed $\mu_j(R, u)$ is (R', u'), where

a) $R' = \mu_j(R)$;

b) $u' = u \setminus \{u_j\} \cup \{u'_j\}$, where u'_j is defined by the exchange relation

$$u_j u'_j = \prod_{\substack{\text{arrows} \\ i \to j}} u_i + \prod_{\substack{\text{arrows} \\ j \to k}} u_k .$$

$$
\begin{tikzcd}
 & & & & x_1 \arrow[r] \arrow[dlll, leftrightarrow, "\mu_1"'] & x_2 \arrow[r] \arrow[d, leftrightarrow, "\mu_2"'] & x_3 \arrow[drrr, leftrightarrow, "\mu_3"] & & & & \\
\frac{1+x_2}{x_1} & x_2 \arrow[l] \arrow[r] \arrow[dl, leftrightarrow, "\mu_2"] \arrow[dr, leftrightarrow, "\mu_3"'] & x_3 & & x_1 \arrow[rr, bend left] & \frac{x_1+x_3}{x_2} \arrow[l] \arrow[dl, leftrightarrow, "\mu_1"] \arrow[dr, leftrightarrow, "\mu_3"'] & x_3 \arrow[l] & & x_1 \arrow[r] & x_2 \arrow[dl, leftrightarrow, "\mu_1"] \arrow[dr, leftrightarrow, "\mu_2"'] & \frac{1+x_2}{x_3} \arrow[l] \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{tikzcd}
$$

Clusters, cluster variables and the cluster algebra

Let Q be a quiver with n vertices.

Definition

a) The initial seed is $(Q, x) = (Q, \{x_1, \ldots, x_n\})$.

b) A cluster is an n-tuple u appearing in a seed (R, u) obtained from (Q, x) by iterated mutation.

c) The cluster variables are the elements of the clusters.

d) The cluster algebra $\mathcal{A}_Q$ is the subalgebra of the field $\mathbb{Q}(x_1, \ldots, x_n)$ generated by the cluster variables.

e) A cluster monomial is a product of powers of cluster variables which all belong to the same cluster.

Remark

If Q is mutation equivalent to Q', then $\mathcal{A}_Q \xrightarrow{\sim} \mathcal{A}_{Q'}$.

Fundamental properties

Let Q be a connected quiver.

Theorem (Fomin-Zelevinsky, 2002)

a) *All cluster variables are Laurent polynomials.*

b) *There is only a finite number of cluster variables iff Q is mutation-equivalent to an orientation $\vec{\Delta}$ of an ADE Dynkin diagram Δ. Then Δ is unique and called the cluster type of Q.*

Examples for a) and b): A_3, D_4.

Positivity Theorem (conj. by Fomin-Zelevinsky, 2002)

All cluster variables are Laurent polynomials with non negative coefficients.

Remark

Proved for quivers by Schiffler–Lee (2013, combinatorics) and for valued quivers by Gross–Hacking–Keel–Kontsevich (2014, tools from mirror symmetry: scattering diagrams and broken lines).

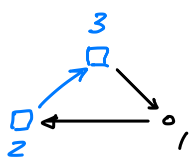
3. Cluster algebras with coefficients

Rk: Coefficients are essential when we want to

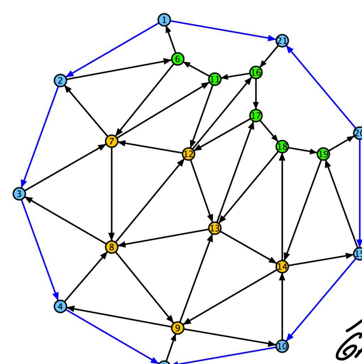
- identify cluster algebras with coord. alg. $\mathbb{C}[X]$ of varieties X.
- "glue" cluster algebras (e.g. in Fock-Goncharov's "amalgamation" construction)
- construct (quasi-) cluster automorphisms of cluster algebras

Def.: An ice quiver is a pair (Q, F) where Q is a finite quiver (w/o loops nor 2-cycles) and $F \subseteq Q$ a subquiver (not necessarily full) called the frozen part.

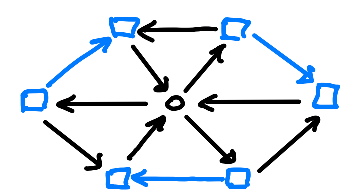
Examples:



$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$$



$\widehat{Gr}(4,9)$



Fock-Goncharov's variety of "triples of flags in $PSL_3(\mathbb{R})$".

Let (Q,F) be an ice quiver and suppose that $Q_0 = \{\underbrace{1,\dots,n}_{\notin F_0}, \underbrace{n+1,\dots,m}_{\in F_0}\}$

Def.: The **cluster variables** for (Q,F) are obtained by iteratively mutating the initial seed $(Q, \{x_1, \dots, x_m\})$ at non frozen vertices.

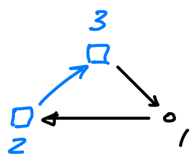
Prop. (FZ '06, Refined Laurent phen.): They lie in $\mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, \overbrace{x_{n+1}, \dots, x_m}^{\text{coefficients = frozen cl. var.}}]$.

Def.: $\mathcal{A}_{Q,F}$ = cluster alg. with non invertible coeff.

$= \mathbb{Q}[x_{n+1}, \dots, x_m]$-subalg. of $\mathbb{Q}(x_1, \dots, x_m)$ gen. by the cluster variables

Rk: The combinatorics of the cluster algebra $\mathcal{A}_{Q,F}$ are "the same" as those of the cluster algebra $\mathcal{A}_{\overline{Q}}$ without coefficients, where $\overline{Q}$ is the non frozen part of Q (Cerulli-K-Labardini-Plamondon '13).

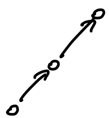
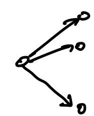
Examples: 1) $N = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, a,b,c \in \mathbb{C} \right\} \subseteq SL_3(\mathbb{C})$.

Q, F : 

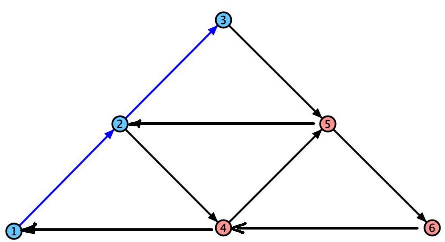
$\Longrightarrow \quad \mathbb{C} \underset{\mathbb{Q}}{\otimes} \mathcal{A}_{Q,F} \xrightarrow{\sim} \mathbb{C}[N]$, $\quad x_1 \longmapsto a$, $\quad x_2 \longmapsto ac - b$, $\quad x_3 \longmapsto b$

Christof Geiss, Jan Schröer, Bernard Leclerc

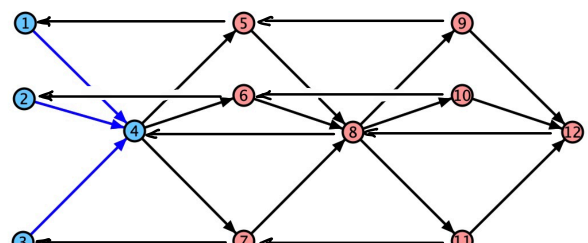
2) Geiss-Leclerc-Schröer '06: More generally, let R be a Dynkin quiver, e.g.

R :  or R :  .

Let $Q = Q_R$ be its "3-CY-completed Auslander-Reiten quiver", e.g.



or



.

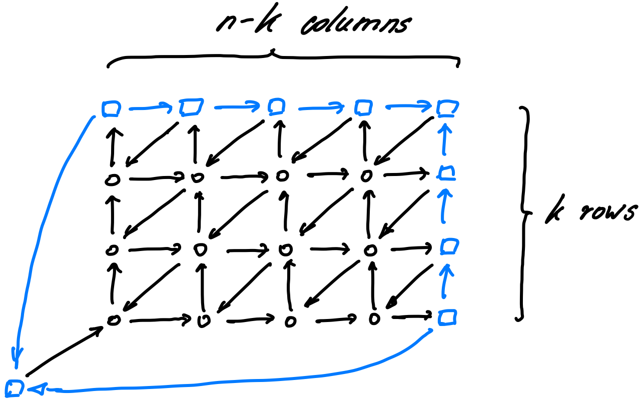
Then there is a can. isomorphism

$$\mathcal{A}_{Q,F} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathbb{C}[N],$$

where $N \leq G$ is any max. unipotent subgroup of the simple alg. group G of the same type as Q.

3) $\widehat{Gr(k,n)}$ = Plücker cone over the Grassmannian of k-dim. subsp. in $\mathbb{C}^n$.

Q, F:



$\widehat{Gr}(4,9)$

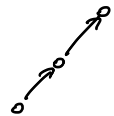
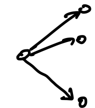
$$\mathbb{C} \otimes_{\mathbb{Q}} \mathcal{A}_{Q,F} \xrightarrow[\text{Scott '06}]{\sim} \mathbb{C}[\widehat{Gr(k,n)}]$$

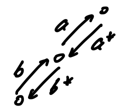
cluster variables $\} \longleftarrow \{$ Plücker coordinates

exchange relations $\} \longleftarrow \{$ Plücker relations

4. An introduction to Higgs categories

4.1 A class of typical examples

Let Q be a Dynkin quiver, e.g. Q:  or Q:  . Let $k=\mathbb{C}$.

Let $\Lambda=\Lambda_Q$ be the *preprojective algebra* of Q, e.g.  , $-b^*b=0$, $bb^*-a^*a=0$, $aa^*=0$.

Let $\operatorname{mod}\Lambda$ be the *abelian cat.* of k-fin.-dim. right Λ-modules.

Let $N\subseteq G$ be any max. unipotent subgroup in the simple alg. group G of the same type as Q.

Thm (Geiss-Leclerc-Schröer '06): There is a canonical explicit *cluster character*

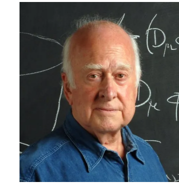
$$\varphi : \operatorname{mod}\Lambda \longrightarrow \mathbb{C}[N] \simeq \mathcal{A}_{\mathbb{Q}_R,F} \underset{\mathbb{Q}}{\otimes} \mathbb{C}$$

inducing a bijection

$\operatorname{Ext}^1(X,X)=0$

$$\left\{\begin{matrix}\text{reachable rigid}\\ \text{indecomposables}\\ \text{of } \operatorname{mod}\Lambda\end{matrix}\right\}/\text{isom.} \xrightarrow{\sim} \left\{\begin{matrix}\text{cluster}\\ \text{variables}\end{matrix}\right\} \subseteq \mathbb{C}[N].$$

Rk: Modules over a preprojective algebra Λ_R are the algebraic analogues of Higgs bundles on a curve. The cat. $\mathrm{mod}\,\Lambda_R$ is a typical example of a Higgs category.


Peter Higgs, 1929-2024

4.2 Higgs categories

Let $Q \supseteq F$ be an ice quiver and W a non degenerate potential on Q (cf. App. A).

Thm (Wu '23, K-Wu): Under mild finiteness assumptions on Q, F, W (cf. App. A), there is a canonical extriangulated category $\mathcal{H}_{Q,F,W}$, the Higgs category, and a can. explicit cluster character (cf. App. B)


Yilin Wu 吴燚林

Nakaoka–Palu '19, e.g. htpy cat. of an exact ∞-cat.

$$CC : \mathcal{H}_{Q,F,W} \longrightarrow \mathcal{A}_{Q,F}$$

actually: $\mathcal{U}_{Q,F}$ = upper cluster alg.

inducing bijections

$\mathrm{Ext}^1(X,X)=0$

$$\left\{ \begin{array}{c} \text{reachable rigid} \\ \text{indecomposables} \\ \text{of } \mathcal{H}_{Q,F,W} \end{array} \right\} / \text{isom.} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{cluster} \\ \text{variables} \end{array} \right\} \subseteq \mathbb{C}[N] \,.$$

$$\{\text{indec. projectives}\} / \text{isom.} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{coefficients} \\ = \text{frozen cl. var.} \end{array} \right\}$$

Examples: 1) In the situation of Geiss-Leclerc-Schröer, i.e. R Dynkin, $Q = Q_R$, $W = W_{can}$, we have

$\mathcal{H}_{Q,F,W_{can}} \simeq \operatorname{mod}\Lambda_R$, where Λ_R is the preprojective algebra of R.

2) If $Q \supseteq F$ is the ice quiver of the Grassmannian $\widehat{Gr}(k,n)$, we find that $\mathcal{H}_{Q,F,W_{can}}$ is

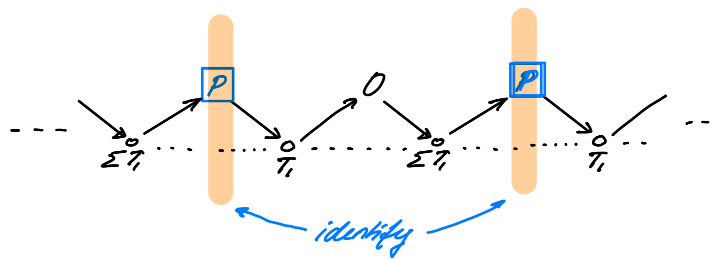
Jensen-King-Su's (2016) Grassmannian cluster category of Cohen-Macaulay modules $\operatorname{cm}(B)$,

Hom-infinite!

$B = \Lambda_{\tilde{A}_{n-1}} / (x^k - y^{n-k})$.

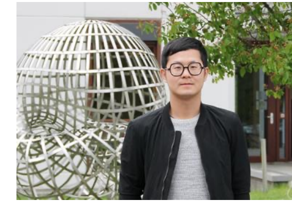
3) For $Q \supseteq F$: $\square \longrightarrow \overset{1}{\circ}$, $W = 0$, the Higgs cat. $\mathcal{H}$ is neither exact nor triangulated (but extriangulated).

Its Auslander-Reiten quiver is



4) For varieties of triples of flags (Fock-Goncharov, Jiarui Fei, Ian Le, ..., Linhui Shen) of type Δ,

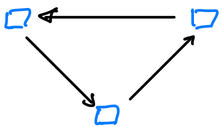
we also obtain strictly extriangulated categories $\mathcal{H}_\Delta$, as shown by Miantao Liu.



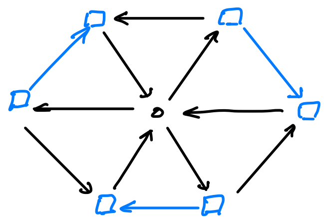
Miantao Liu 刘绵涛

Some more examples of the corresponding ice quivers:

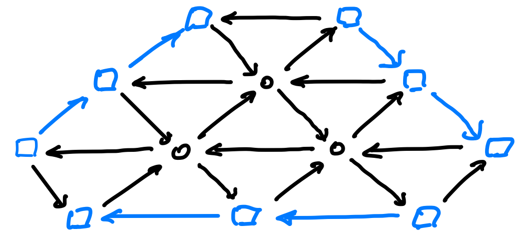
$\Delta = A_1$



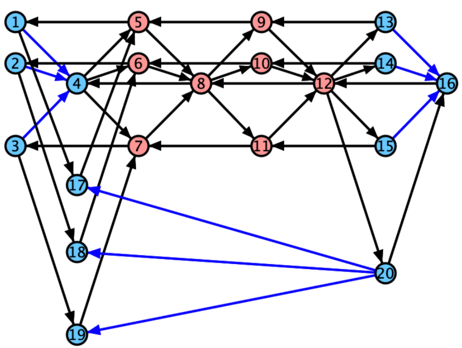
$\Delta = A_2$



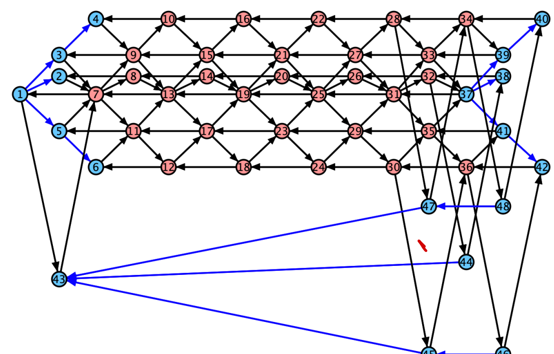
$\Delta = A_3$



$\Delta = D_4$



$\Delta = E_6$



4.3 Application: M. Christ's categorification of higher Teichmüller spaces

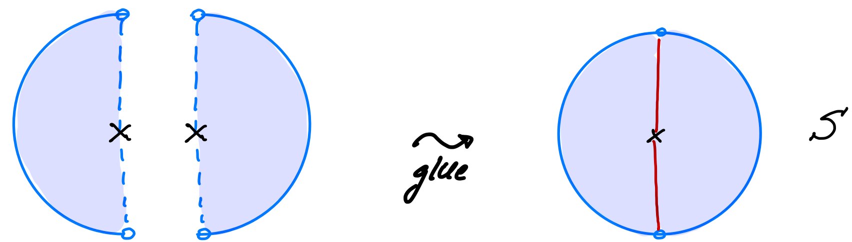
Thm (Goncharov–Shen '19):

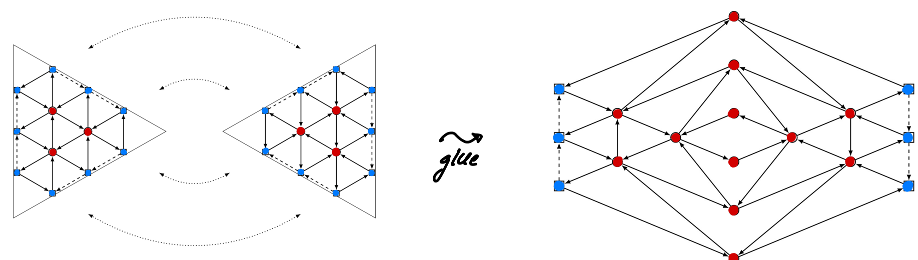
By glueing varieties of triples of flags of Dynkin type Δ following a triangulation of a marked decorated surface S, we obtain the higher Teichmüller space $\mathcal{H}\mathcal{T}_{S,\Delta}$.

Thm (Merlin Christ '25):

Analogously, by glueing copies of Miantao Liu's category $\mathcal{H}_\Delta$, we obtain an additive categorification of $\mathcal{H}\mathcal{T}_{S,\Delta}$.

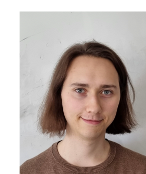
Example:






Alexander Goncharov


沈临辉 Linhui Shen


Merlin Christ

4.4 Construction of the Higgs category

$k=\mathbb{C}$, (Q,F) an ice quiver (Q finite, w/o loops nor 2-cycles)

kQ = (completed) path algebra

assumed "non-degenerate" e.g. "generic"

$W \in HC_0(kQ)$ a potential (= formal linear comb. of oriented cycles in Q)

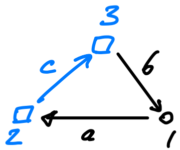
Associated Ginzburg morphism

$$\gamma : \Pi_2(kF) \longrightarrow \Pi_3(kQ, kF, W)$$

with rel. left 3-CY-structure (Yeung '16).

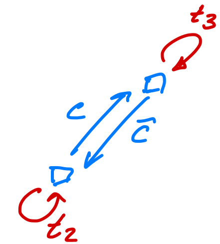
Victor Ginzburg
(Chicago)

Example:



, $W = abc$

2-dim. Ginzburg algebra $\longrightarrow$ $\Pi_2(kF)$:



$|c| = |\bar c| = 0$

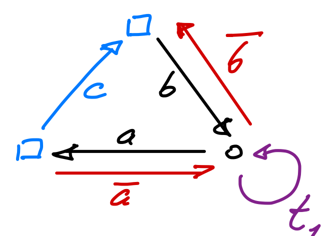
$|t_i| = -1$

$dt_2 = -\bar c c\,, \quad dt_1 = c\bar c$

$$\Pi_2(kF) \xrightarrow{\ \gamma\ } \Pi_3(kQ, kF, W)$$

$$\bar c \longmapsto \partial_c W = ab$$

3-dim. rel. Ginzburg alg. $\longrightarrow$ $\Pi_3(kQ, kF, W)$



$|t_1| = -2$

$|\bar a| = |\bar b| = -1$

$d\bar a = \partial_a W = bc$

$d\bar b = \partial_b W = ca$

Nontrivial exercise: Show that $H^* \Pi_3(\ldots)$ is conc. in degree 0.

Abbreviation: $\Pi_2(kF) \longrightarrow \Pi_3(kQ, kF, W)$

$$\begin{array}{ccc} \Pi_2(kF) & \longrightarrow & \Pi_3(kQ,kF,W) \\ \| & & \| \\ \Pi & & \Gamma \end{array}$$

$\mathcal{D}\Gamma = \{\text{right dg } \Gamma\text{-modules}\}[\text{Qis}^{-1}]$ = (unbounded) derived cat. of Γ

$\cup\!|$

$\mathrm{per}\,\Gamma = (\mathcal{D}\Gamma)^c = \mathrm{thick}(\Gamma_\Gamma)$ = perfect derived cat. of Γ

ψ

S_i = simple 1-dim. dg Γ-module conc. at $i \in Q_0$

$\mathrm{add}\,\Gamma$ = closure under finite coprod. and retracts of Γ_Γ in $\mathcal{D}\Gamma$

$e_F = \sum_{i \in F_0} e_i$ (e_i = path of length 0 at $i \in Q_0$).

Put

$\mathcal{P} = \mathrm{add}(e_F\Gamma) \subseteq \mathrm{per}\,\Gamma$.

From now on, we make the following finiteness assumptions:

1) $\mathcal{P}$ is functorially finite in $\operatorname{add}\Gamma$;

2) $\dim H^0 \Pi_3(\overline{Q}, \overline{W}) < \infty$ remove the frozen part in (Q, W)

Def.: $\mathcal{C}_\Gamma$ = relative cluster cat. $= \operatorname{per}(\Gamma) / \operatorname{thick}(S_i \mid i \notin F_0)$

$\cup\!\mid$ full

$\mathcal{H}_\Gamma$ = Higgs category $= \{ M \in \mathcal{C}_\Gamma \mid \operatorname{Ext}^i(M, P) = 0 = \operatorname{Ext}^i(P, M),\ \forall P \in \mathcal{P},\ \forall i > 0 \}$

Prop.: a) $\mathcal{H}_\Gamma \subseteq \mathcal{C}_\Gamma$ is closed under extensions hence extriangulated

b) For $X, Y \in \mathcal{H}_\Gamma$: $\operatorname{Ext}^1(X, Y)^* \simeq \operatorname{Ext}^1(Y, X)$ ("weak 2-CY-property").

4.5 Cluster characters

Let $\mathcal{H}$ be an Ext^1-finite extriangulated category satisfying the weak 2-CY property.

Def. (Palu): A cluster character on $\mathcal{H}$ is a map $\chi : \mathcal{H} \to R$ to a com. ring R s.th.

1) $X \cong X' \Rightarrow \chi(X) = \chi(X')$;

2) $\chi(X \oplus Y) = \chi(X)\chi(Y)$;

3) For $\dim \mathrm{Ext}^1(X,Y) = 1$ ($= \dim \mathrm{Ext}^1(Y,X)$), we have

$$\chi(X)\chi(Y) = \chi(E) + \chi(E') \qquad (*)$$

where E, E' are given by triangles

$$X \longrightarrow E \longrightarrow Y \xrightarrow{\neq 0} \Sigma X, \quad Y \longrightarrow E' \longrightarrow X \xrightarrow{\neq 0} \Sigma Y$$

Rk: $(*)$ generalizes the exchange relation:

$$x_j x_j' = \prod_{i \to j} x_i + \prod_{j \to k} x_k .$$